

Probability

Part 2: Set Theory and Event Algebra

Exactly one · at least one · three events · odds · independent events

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Types of Events

Impossible / sure

$P(E) = 0$ for impossible, $P(E) = 1$ for sure ($E = S$).

Mutually exclusive

Cannot happen together: $A \cap B = \varnothing$, so $P(A \cup B) = P(A) + P(B)$.

Exhaustive

Together they cover the sample space: $A \cup B = S$.

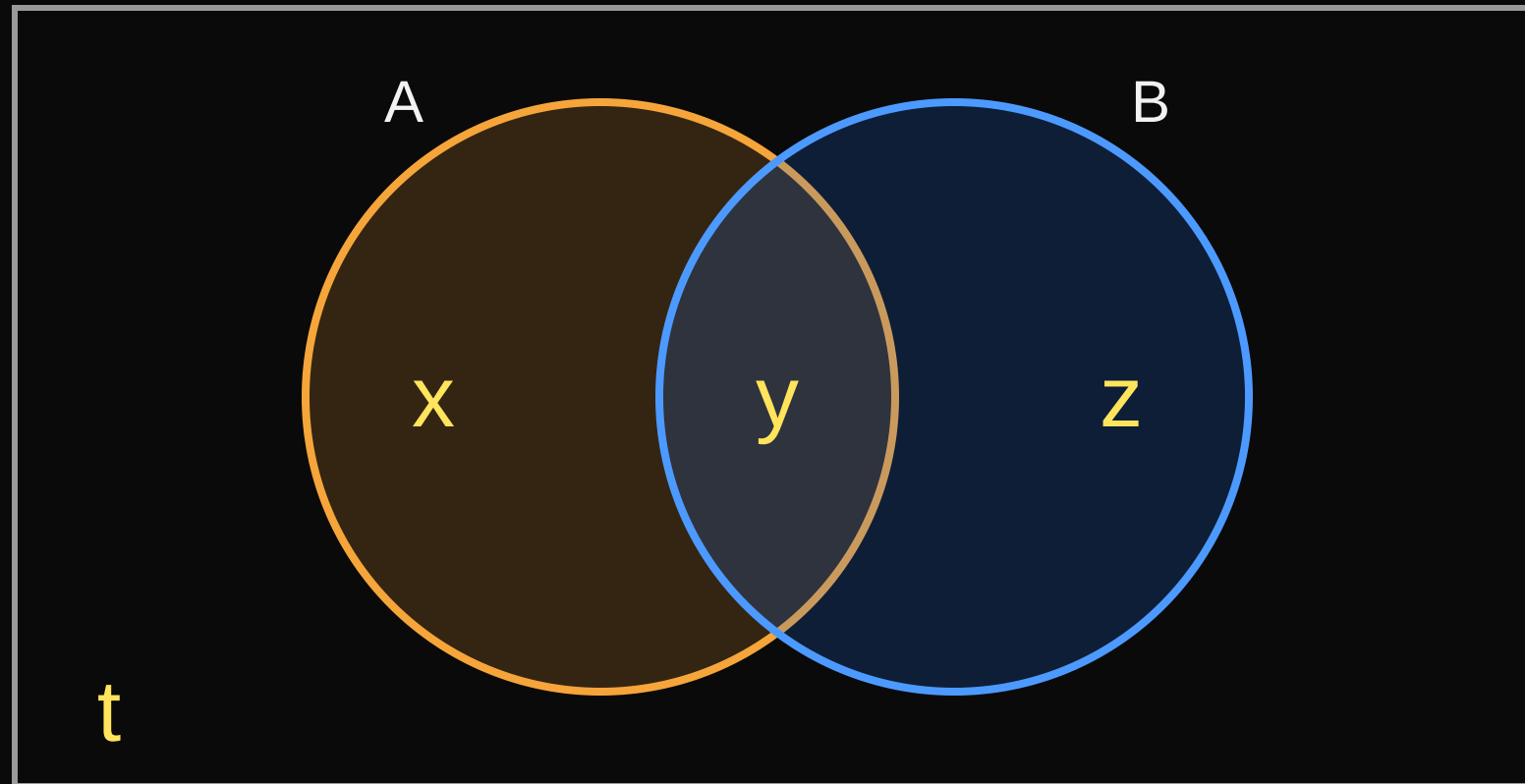
Complementary

$\bar{A}$: $A \cup \bar{A} = S$ and $A \cap \bar{A} = \varnothing$, so $P(\bar{A}) = 1 - P(A)$.

Independent

One does not affect the other: $P(A \cap B) = P(A) \cdot P(B)$.

Two Events: Venn Diagram



Let the four regions have probabilities:

$$x = P(\text{A only}) = P(A \cap \bar{B})$$

$$y = P(A \cap B)$$

$$z = P(\text{B only}) = P(\bar{A} \cap B)$$

$$t = P(\text{neither}) = P(\bar{A} \cap \bar{B})$$

$$x + y + z + t = 1$$

$$P(A) = x + y \quad \cdot \quad P(B) = y + z \quad \cdot \quad P(A \cup B) = x + y + z = P(A) + P(B) - P(A \cap B)$$

Two Events: Formulas

Exactly one occurs

$$x + z = P(A) + P(B) - 2 \cdot P(A \cap B)$$

At least one / neither

$$P(A \cup B) = x + y + z \quad \cdot \quad P(\text{neither}) = 1 - P(A \cup B)$$

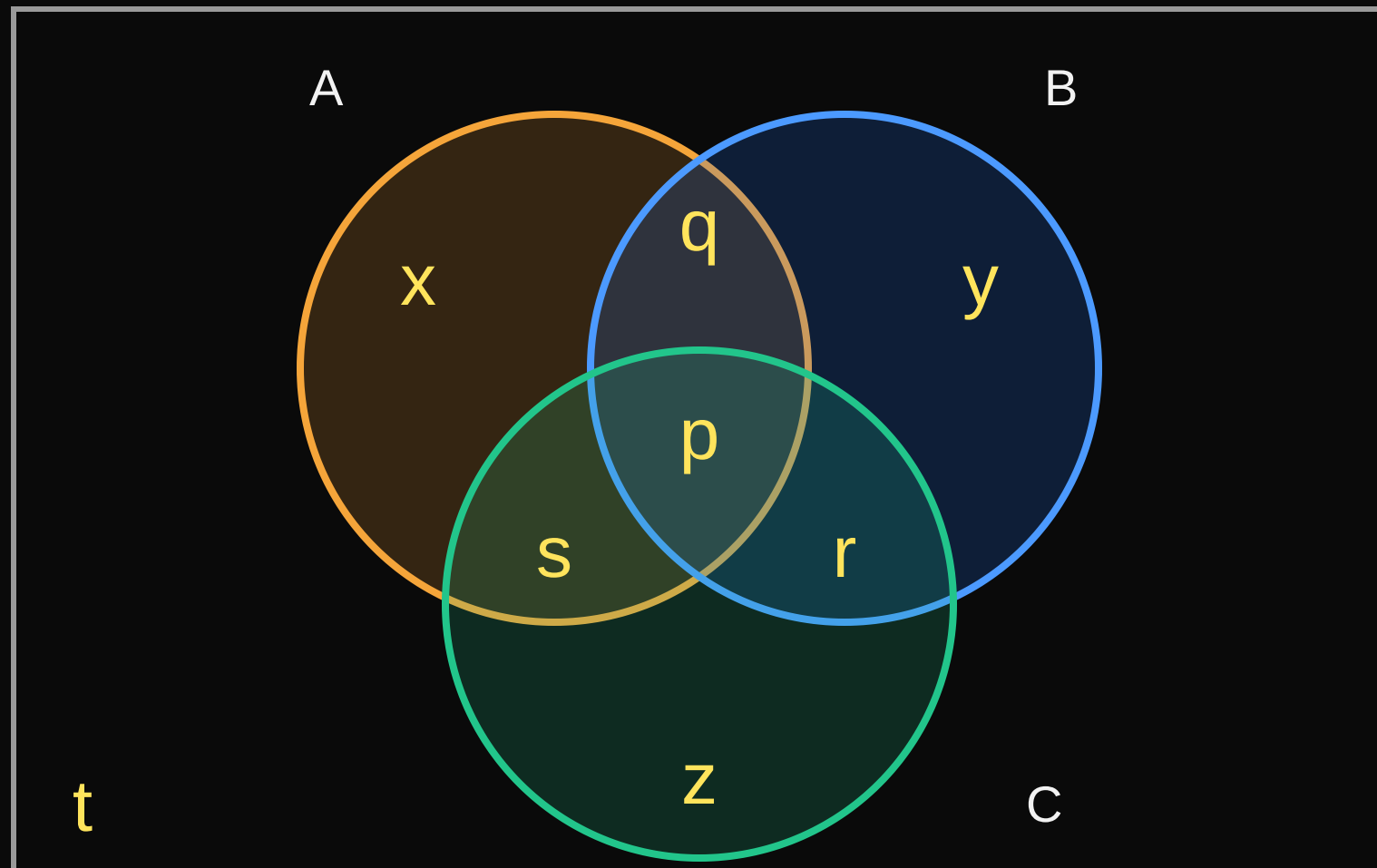
De Morgan

$$P(\bar{A} \cap \bar{B}) = P(A \cup B)' = 1 - P(A \cup B) \quad \cdot \quad P(\bar{A} \cup \bar{B}) = P(A \cap B)' = 1 - P(A \cap B)$$

Only A, only B

$$P(A \cap \bar{B}) = P(A) - P(A \cap B) \quad \cdot \quad P(\bar{A} \cap B) = P(B) - P(A \cap B)$$

Three Events: Venn Diagram



x, y, z = only A, only B, only C

q = $A \cap B$ only · **r** = $B \cap C$ only

s = $A \cap C$ only · **p** = $A \cap B \cap C$

t = none of the three

Exactly one = $x + y + z$

Exactly two = $q + r + s$

At least two = $q + r + s + p$

At least one = $1 - t$

$x + y + z + q + r + s + p + t = 1$

Three Events: Formulas

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

Exactly one

$$\Sigma P(A) - 2 \cdot \Sigma P(A \cap B) + 3 \cdot P(A \cap B \cap C)$$

Exactly two

$$\Sigma P(A \cap B) - 3 \cdot P(A \cap B \cap C)$$

At least two

$$\Sigma P(A \cap B) - 2 \cdot P(A \cap B \cap C)$$

None

$$P(\bar{A} \cap \bar{B} \cap \bar{C}) = 1 - P(A \cup B \cup C)$$

Independent Events and Odds

Independent: $P(A \cap B) = P(A) \cdot P(B)$. Then A and $\bar{B}$, $\bar{A}$ and B , $\bar{A}$ and $\bar{B}$ are also independent.

Three events are independent only if all pairs and the triple multiply: $P(A \cap B \cap C) = P(A)P(B)P(C)$.

Useful for "exactly one"

$$P(\text{exactly one}) = P(A)(1 - P(B)) + (1 - P(A))P(B)$$

At least one

$$P(A \cup B) = 1 - (1 - P(A))(1 - P(B))$$

Odds

Odds in favour $a : b$ means $P(E) = a/(a + b)$. Odds against $a : b$ means $P(E) = b/(a + b)$.

Questions Based on Set Theory

Part 3 · Exactly one, three events, independent events

5 questions · JEE Main + JEE Advanced (archive)

Part 3 · Q.1

Let A and B be two events such that the probability that **exactly one** of them occurs is $\frac{2}{5}$ and the probability that **A or B** occurs is $\frac{1}{2}$. Then the probability that **both** of them occur together is

- A.** 0.02 **B.** 0.20 **C.** 0.01 **D.** 0.10

JEE Main 2020 (8 Jan Shift 2)

Q.1 · Solution

Let $P(A \cap B) = y$.

Exactly one: $P(A) + P(B) - 2y = 2/5$

A or B: $P(A) + P(B) - y = 1/2$

Subtract: $(P(A) + P(B) - y) - (P(A) + P(B) - 2y) = y = 1/2 - 2/5 = \mathbf{1/10 = 0.10}$

Answer: (D)

Exactly one + both = at least one: $2/5 + y = 1/2$.

Part 3 · Q.2

The probability of happening **at least one** of the events A and B is 0.6. If A and B happen **simultaneously** with probability 0.2, then $P(\bar{A}) + P(\bar{B}) =$

- A.** 0.4 **B.** 0.8 **C.** 1.2 **D.** 1.4

IIT-JEE 1987 · also MHT CET 2021 & 2022

Q.2 · Solution

$$P(A \cup B) = 0.6 \text{ and } P(A \cap B) = 0.2$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B), \text{ so } P(A) + P(B) = 0.6 + 0.2 = 0.8$$

$$P(\bar{A}) + P(\bar{B}) = (1 - P(A)) + (1 - P(B)) = 2 - 0.8 = \mathbf{1.2}$$

Answer: (C)

Add the two complements: 2 minus $(P(A) + P(B))$.

Part 3 · Q.3

The probabilities of three events A, B and C are $P(A) = 0.6$, $P(B) = 0.4$, $P(C) = 0.5$. If $P(A \cup B) = 0.8$, $P(A \cap C) = 0.3$, $P(A \cap B \cap C) = 0.2$, $P(B \cap C) = \beta$ and $P(A \cup B \cup C) = \alpha$, where $0.85 \leq \alpha \leq 0.95$, then β lies in the interval

- A.** $[0.35, 0.36]$ **B.** $[0.25, 0.35]$ **C.** $[0.20, 0.25]$ **D.** $[0.36, 0.40]$

JEE Main 2020 (Shift 2)

Q.3 · Solution

$$P(A \cap B) = 0.6 + 0.4 - 0.8 = 0.2$$

$$\text{Three-event formula: } \alpha = 0.6 + 0.4 + 0.5 - 0.2 - \beta - 0.3 + 0.2 = 1.2 - \beta$$

$$\text{So } \beta = 1.2 - \alpha$$

$$0.85 \leq \alpha \leq 0.95 \text{ gives } \mathbf{0.25 \leq \beta \leq 0.35}$$

Answer: (B)

$\beta = 1.2 - \alpha$. Flip the inequality limits.

Part 3 · Q.4

If $P(B) = 3/4$, $P(A \cap B \cap \bar{C}) = 1/3$ and $P(\bar{A} \cap B \cap \bar{C}) = 1/3$, then $P(B \cap C)$ is

A. $1/12$ **B.** $1/6$ **C.** $1/15$ **D.** $1/9$

IIT-JEE 2003

Q.4 · Solution

B is split by C: $P(B) = P(B \cap C) + P(B \cap \bar{C})$

$B \cap \bar{C}$ is split by A: $P(B \cap \bar{C}) = P(A \cap B \cap \bar{C}) + P(\bar{A} \cap B \cap \bar{C}) = 1/3 + 1/3 = 2/3$

$P(B \cap C) = 3/4 - 2/3 = \mathbf{1/12}$

Answer: (A)

$P(B) = (B \text{ inside } C) + (B \text{ outside } C)$. Subtract.

Part 3 · Q.5

If A and B are two **independent** events such that $P(A \cap \bar{B}) = 3/25$ and $P(\bar{A} \cap B) = 8/25$, then $P(A) =$

A. $1/5$ **B.** $3/8$ **C.** $2/5$ **D.** $4/5$

IIT-JEE 2002

Q.5 · Solution

Independent: $a(1 - b) = 3/25$ and $(1 - a)b = 8/25$

Subtract: $a - ab - b + ab = a - b = 3/25 - 8/25 = -1/5$, so $b = a + 1/5$

Put in the first: $a(4/5 - a) = 3/25$, i.e. $25a^2 - 20a + 3 = 0$, so $a = 1/5$ or $3/5$

Only $1/5$ is among the options, with $b = 2/5$ ($a = 3/5$, $b = 4/5$ also fits). **$P(A) = 1/5$**

Answer: (A)

Two equations, two unknowns. Test the options: $1/5$ and $2/5$ work.