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SRI VIDYA ARADHANA ACADEMY

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PCMB — PHYSICS CHEMISTRY MATHEMATICS BIOLOGY

NEET/JEE

Topic: Trigonometric Identities

Sub: Mathematics

Solutions to Assignment: 02

Prof. Chetan Sir

1. $\sin\theta\cot\theta = \cos\theta$

$$\begin{aligned} \text{LHS} &= \sin\theta\cot\theta \\ &= \sin\theta\left(\frac{\cos\theta}{\sin\theta}\right) \quad [\text{Using } \cot\theta = \frac{\cos\theta}{\sin\theta}] \\ &= \cos\theta \quad [\text{Simplifying}] \\ &= \text{RHS} \end{aligned}$$

■

2. $\cos\theta\tan\theta = \sin\theta$

$$\begin{aligned} \text{LHS} &= \cos\theta\tan\theta \\ &= \cos\theta\left(\frac{\sin\theta}{\cos\theta}\right) \quad [\text{Using } \tan\theta = \frac{\sin\theta}{\cos\theta}] \\ &= \sin\theta \quad [\text{Simplifying}] \\ &= \text{RHS} \end{aligned}$$

■

3. $\sec\theta\cot\theta = \operatorname{cosec}\theta$

$$\begin{aligned} \text{LHS} &= \sec\theta\cot\theta \\ &= \left(\frac{1}{\cos\theta}\right)\left(\frac{\cos\theta}{\sin\theta}\right) \quad [\text{Using definitions}] \\ &= \frac{1}{\sin\theta} \quad [\text{Simplifying}] \\ &= \operatorname{cosec}\theta \quad [\text{Using } \operatorname{cosec}\theta = \frac{1}{\sin\theta}] \\ &= \text{RHS} \end{aligned}$$

■

4. $\operatorname{cosec}\theta\tan\theta = \sec\theta$

$$\begin{aligned} \text{LHS} &= \operatorname{cosec}\theta\tan\theta \\ &= \left(\frac{1}{\sin\theta}\right)\left(\frac{\sin\theta}{\cos\theta}\right) \quad [\text{Using definitions}] \\ &= \frac{1}{\cos\theta} \quad [\text{Simplifying}] \\ &= \sec\theta \quad [\text{Using } \sec\theta = \frac{1}{\cos\theta}] \\ &= \text{RHS} \end{aligned}$$

■

5. $(1-\sin^2\theta)\sec^2\theta = 1$

$$\begin{aligned} \text{LHS} &= (1-\sin^2\theta)\sec^2\theta \\ &= (\cos^2\theta)\sec^2\theta \quad [\text{Pythagorean Id: } \sin^2\theta+\cos^2\theta = 1] \\ &= \cos^2\theta\left(\frac{1}{\cos^2\theta}\right) \quad [\text{Using } \sec\theta = \frac{1}{\cos\theta}] \\ &= 1 \quad [\text{Simplifying}] \\ &= \text{RHS} \end{aligned}$$

■

6. $(1-\cos^2\theta)\operatorname{cosec}^2\theta = 1$

$$\begin{aligned}
\text{LHS} &= (1-\cos^2\theta)\operatorname{cosec}^2\theta \\
&= (\sin^2\theta)\operatorname{cosec}^2\theta \quad [\text{Pythagorean Id: } \sin^2\theta+\cos^2\theta=1] \\
&= \sin^2\theta\left(\frac{1}{\sin^2\theta}\right) \quad [\text{Using } \operatorname{cosec}\theta = \frac{1}{\sin\theta}] \\
&= 1 \quad [\text{Simplifying}] \\
&= \text{RHS}
\end{aligned}$$

■

7. $\tan^2\theta\cos^2\theta = \sin^2\theta$

$$\begin{aligned}
\text{LHS} &= \tan^2\theta\cos^2\theta \\
&= \left(\frac{\sin^2\theta}{\cos^2\theta}\right)\cos^2\theta \quad [\text{Using } \tan\theta = \frac{\sin\theta}{\cos\theta}] \\
&= \sin^2\theta \quad [\text{Simplifying}] \\
&= \text{RHS}
\end{aligned}$$

■

8. $(\sec^2\theta-1)\cot^2\theta = 1$

$$\begin{aligned}
\text{LHS} &= (\sec^2\theta-1)\cot^2\theta \\
&= (\tan^2\theta)\cot^2\theta \quad [\text{Pythagorean Id: } 1+\tan^2\theta=\sec^2\theta] \\
&= \tan^2\theta\left(\frac{1}{\tan^2\theta}\right) \quad [\text{Using } \cot\theta = \frac{1}{\tan\theta}] \\
&= 1 \quad [\text{Simplifying}] \\
&= \text{RHS}
\end{aligned}$$

■

9. $(\operatorname{cosec}^2\theta-1)\tan^2\theta = 1$

$$\begin{aligned}
\text{LHS} &= (\operatorname{cosec}^2\theta-1)\tan^2\theta \\
&= (\cot^2\theta)\tan^2\theta \quad [\text{Pythagorean Id: } 1+\cot^2\theta=\operatorname{cosec}^2\theta] \\
&= \cot^2\theta\left(\frac{1}{\cot^2\theta}\right) \quad [\text{Using } \tan\theta = \frac{1}{\cot\theta}] \\
&= 1 \quad [\text{Simplifying}] \\
&= \text{RHS}
\end{aligned}$$

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10. $(1+\cot^2 A)\sin^2 A = 1$

$$\begin{aligned}
\text{LHS} &= (1+\cot^2 A)\sin^2 A \\
&= (\operatorname{cosec}^2 A)\sin^2 A \quad [\text{Pythagorean Id: } 1+\cot^2 A=\operatorname{cosec}^2 A] \\
&= \left(\frac{1}{\sin^2 A}\right)\sin^2 A \quad [\text{Using } \operatorname{cosec} A = \frac{1}{\sin A}] \\
&= 1 \quad [\text{Simplifying}] \\
&= \text{RHS}
\end{aligned}$$

■

11. $\sin^2 x\sec^2 x = \tan^2 x$

$$\begin{aligned}
\text{LHS} &= \sin^2 x\sec^2 x \\
&= \sin^2 x\left(\frac{1}{\cos^2 x}\right) \quad [\text{Using } \sec x = \frac{1}{\cos x}] \\
&= \frac{\sin^2 x}{\cos^2 x} \\
&= \tan^2 x \quad [\text{Using } \tan x = \frac{\sin x}{\cos x}] \\
&= \text{RHS}
\end{aligned}$$

■

12. $\cos^2 x \operatorname{cosec}^2 x = \cot^2 x$

$$\begin{aligned}
\text{LHS} &= \cos^2 x \operatorname{cosec}^2 x \\
&= \cos^2 x \left(\frac{1}{\sin^2 x}\right) \quad [\text{Using } \operatorname{cosec} x = \frac{1}{\sin x}] \\
&= \frac{\cos^2 x}{\sin^2 x} \\
&= \cot^2 x \quad [\text{Using } \cot x = \frac{\cos x}{\sin x}] \\
&= \text{RHS}
\end{aligned}$$
■

13. $\cos\theta(\sec\theta - \cos\theta) = \sin^2\theta$

$$\begin{aligned}
\text{LHS} &= \cos\theta(\sec\theta - \cos\theta) \\
&= \cos\theta\sec\theta - \cos^2\theta \quad [\text{Distributing}] \\
&= \cos\theta\left(\frac{1}{\cos\theta}\right) - \cos^2\theta \quad [\text{Using } \sec\theta = \frac{1}{\cos\theta}] \\
&= 1 - \cos^2\theta \quad [\text{Simplifying}] \\
&= \sin^2\theta \quad [\text{Pythagorean Id: } \sin^2\theta + \cos^2\theta = 1] \\
&= \text{RHS}
\end{aligned}$$
■

14. $\tan\theta(\cot\theta + \tan\theta) = \sec^2\theta$

$$\begin{aligned}
\text{LHS} &= \tan\theta(\cot\theta + \tan\theta) \\
&= \tan\theta\cot\theta + \tan^2\theta \quad [\text{Distributing}] \\
&= \tan\theta\left(\frac{1}{\tan\theta}\right) + \tan^2\theta \quad [\text{Using } \cot\theta = \frac{1}{\tan\theta}] \\
&= 1 + \tan^2\theta \quad [\text{Simplifying}] \\
&= \sec^2\theta \quad [\text{Pythagorean Id: } 1 + \tan^2\theta = \sec^2\theta] \\
&= \text{RHS}
\end{aligned}$$
■

15. $\sin^2\theta\cot^2\theta + \sin^2\theta = 1$

$$\begin{aligned}
\text{LHS} &= \sin^2\theta\cot^2\theta + \sin^2\theta \\
&= \sin^2\theta\left(\frac{\cos^2\theta}{\sin^2\theta}\right) + \sin^2\theta \quad [\text{Using } \cot\theta = \frac{\cos\theta}{\sin\theta}] \\
&= \cos^2\theta + \sin^2\theta \quad [\text{Simplifying}] \\
&= 1 \quad [\text{Pythagorean Id: } \sin^2\theta + \cos^2\theta = 1] \\
&= \text{RHS}
\end{aligned}$$
■

16. $\frac{1-\cos^2\theta}{\sin\theta} = \sin\theta$

$$\begin{aligned}
\text{LHS (corrected)} &= \frac{1-\cos^2\theta}{\sin\theta} \\
&= \frac{\sin^2\theta}{\sin\theta} \quad [\text{Pythagorean Id}] \\
&= \sin\theta \\
&= \text{RHS}
\end{aligned}$$

17. $\frac{(1+\tan^2\theta)\cot\theta}{\csc^2\theta} = \tan\theta$

$$\begin{aligned}
\text{LHS} &= \frac{(1+\tan^2\theta)\cot\theta}{\operatorname{cosec}^2\theta} \\
&= \frac{(\sec^2\theta)\cot\theta}{\operatorname{cosec}^2\theta} \quad \text{[Pythagorean Id]} \\
&= \frac{\left(\frac{1}{\cos^2\theta}\right)\left(\frac{\cos\theta}{\sin\theta}\right)}{\left(\frac{1}{\sin^2\theta}\right)} \quad \text{[Definitions]} \\
&= \frac{\frac{1}{\cos\theta\sin\theta}}{\frac{1}{\sin^2\theta}} \\
&= \frac{1}{\cos\theta\sin\theta}\cdot\frac{\sin^2\theta}{1} \quad \text{[Fraction division]} \\
&= \frac{\sin\theta}{\cos\theta} \quad \text{[Simplifying]} \\
&= \tan\theta \quad \text{[Definition]} \\
&= \text{RHS}
\end{aligned}$$

■

18. $\frac{\sec^2\theta}{1+\cot^2\theta} = \tan^2\theta$

$$\begin{aligned}
\text{LHS} &= \frac{\sec^2\theta}{1+\cot^2\theta} \\
&= \frac{\sec^2\theta}{\operatorname{cosec}^2\theta} \quad \text{[Pythagorean Id]} \\
&= \frac{1/\cos^2\theta}{1/\sin^2\theta} \quad \text{[Definitions]} \\
&= \frac{1}{\cos^2\theta}\cdot\frac{\sin^2\theta}{1} \\
&= \frac{\sin^2\theta}{\cos^2\theta} \quad \text{[Simplifying]} \\
&= \tan^2\theta \quad \text{[Definition]} \\
&= \text{RHS}
\end{aligned}$$

■

19. $\frac{\operatorname{cosec}^2\theta}{1+\tan^2\theta} = \cot^2\theta$

$$\begin{aligned}
\text{LHS} &= \frac{\operatorname{cosec}^2\theta}{1+\tan^2\theta} \\
&= \frac{\operatorname{cosec}^2\theta}{\sec^2\theta} \quad \text{[Pythagorean Id]} \\
&= \frac{1/\sin^2\theta}{1/\cos^2\theta} \quad \text{[Definitions]} \\
&= \frac{1}{\sin^2\theta}\cdot\frac{\cos^2\theta}{1} \\
&= \frac{\cos^2\theta}{\sin^2\theta} \quad \text{[Simplifying]} \\
&= \cot^2\theta \quad \text{[Definition]} \\
&= \text{RHS}
\end{aligned}$$

■

20. $\cos^2 A + \frac{1}{1+\cot^2 A} = 1$

$$\begin{aligned}
\text{LHS} &= \cos^2 A + \frac{1}{1+\cot^2 A} \\
&= \cos^2 A + \frac{1}{\operatorname{cosec}^2 A} \quad \text{[Pythagorean Id]} \\
&= \cos^2 A + \sin^2 A \quad \text{[Using } \sin A = \frac{1}{\operatorname{cosec} A}\text{]} \\
&= 1 \quad \text{[Pythagorean Id]} \\
&= \text{RHS}
\end{aligned}$$

■

21. $\sin^2 A + \frac{1}{1+\tan^2 A} = 1$

$$\begin{aligned}\text{LHS} &= \sin^2 A + \frac{1}{1+\tan^2 A} \\ &= \sin^2 A + \frac{1}{\sec^2 A} \quad \text{[Pythagorean Id]} \\ &= \sin^2 A + \cos^2 A \quad [\text{Using } \cos A = \frac{1}{\sec A}] \\ &= 1 \quad \text{[Pythagorean Id]} \\ &= \text{RHS}\end{aligned}$$

■

22. $\frac{\tan^2 A}{1+\tan^2 A} + \frac{\cot^2 A}{1+\cot^2 A} = 1$

$$\begin{aligned}\text{LHS} &= \frac{\tan^2 A}{1+\tan^2 A} + \frac{\cot^2 A}{1+\cot^2 A} \\ &= \frac{\tan^2 A}{\sec^2 A} + \frac{\cot^2 A}{\text{cosec}^2 A} \quad \text{[Pythagorean Ids]} \\ &= \frac{\sin^2 A/\cos^2 A}{1/\cos^2 A} + \frac{\cos^2 A/\sin^2 A}{1/\sin^2 A} \quad \text{[Definitions]} \\ &= \sin^2 A + \cos^2 A \quad \text{[Simplifying fractions]} \\ &= 1 \quad \text{[Pythagorean Id]} \\ &= \text{RHS}\end{aligned}$$

■

23. $\frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta}$

$$\begin{aligned}\text{LHS} &= \frac{1-\cos\theta}{\sin\theta} \\ &= \frac{(1-\cos\theta)(1+\cos\theta)}{\sin\theta(1+\cos\theta)} \quad \text{[Multiply by conjugate]} \\ &= \frac{1-\cos^2\theta}{\sin\theta(1+\cos\theta)} \quad \text{[Difference of squares]} \\ &= \frac{\sin^2\theta}{\sin\theta(1+\cos\theta)} \quad \text{[Pythagorean Id]} \\ &= \frac{\sin\theta}{1+\cos\theta} \quad \text{[Simplifying]} \\ &= \text{RHS}\end{aligned}$$

■

24. $\frac{\cos\theta}{1-\sin\theta} = \frac{1+\sin\theta}{\cos\theta}$

$$\begin{aligned}\text{LHS} &= \frac{\cos\theta}{1-\sin\theta} \\ &= \frac{\cos\theta(1+\sin\theta)}{(1-\sin\theta)(1+\sin\theta)} \quad \text{[Multiply by conjugate]} \\ &= \frac{\cos\theta(1+\sin\theta)}{1-\sin^2\theta} \quad \text{[Difference of squares]} \\ &= \frac{\cos\theta(1+\sin\theta)}{\cos^2\theta} \quad \text{[Pythagorean Id]} \\ &= \frac{1+\sin\theta}{\cos\theta} \quad \text{[Simplifying]} \\ &= \text{RHS}\end{aligned}$$

■

25. $\frac{\cos\theta}{1+\sin\theta} = \frac{1-\sin\theta}{\cos\theta}$

$$\begin{aligned}
\text{LHS} &= \frac{\cos\theta}{1+\sin\theta} \\
&= \frac{\cos\theta(1-\sin\theta)}{(1+\sin\theta)(1-\sin\theta)} \quad \text{[Multiply by conjugate]} \\
&= \frac{\cos\theta(1-\sin\theta)}{1-\sin^2\theta} \quad \text{[Difference of squares]} \\
&= \frac{\cos\theta(1-\sin\theta)}{\cos^2\theta} \quad \text{[Pythagorean Id]} \\
&= \frac{1-\sin\theta}{\cos\theta} \quad \text{[Simplifying]} \\
&= \text{RHS}
\end{aligned}$$

■

26. $\frac{\sin\theta}{1-\cos\theta} = \operatorname{cosec}\theta + \cot\theta$

$$\begin{aligned}
\text{LHS} &= \frac{\sin\theta}{1-\cos\theta} \\
&= \frac{\sin\theta(1+\cos\theta)}{(1-\cos\theta)(1+\cos\theta)} \quad \text{[Multiply by conjugate]} \\
&= \frac{\sin\theta(1+\cos\theta)}{1-\cos^2\theta} \quad \text{[Difference of squares]} \\
&= \frac{\sin\theta(1+\cos\theta)}{\sin^2\theta} \quad \text{[Pythagorean Id]} \\
&= \frac{1+\cos\theta}{\sin\theta} \quad \text{[Simplifying]} \\
&= \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta} \quad \text{[Split fraction]} \\
&= \operatorname{cosec}\theta + \cot\theta \quad \text{[Definitions]} \\
&= \text{RHS}
\end{aligned}$$

■

27. $\frac{1+\cos A}{\sin A} = \frac{\sin A}{1-\cos A}$

$$\begin{aligned}
\text{LHS} &= \frac{1+\cos A}{\sin A} \\
&= \frac{(1+\cos A)(1-\cos A)}{\sin A(1-\cos A)} \quad \text{[Multiply by conjugate]} \\
&= \frac{1-\cos^2 A}{\sin A(1-\cos A)} \quad \text{[Difference of squares]} \\
&= \frac{\sin^2 A}{\sin A(1-\cos A)} \quad \text{[Pythagorean Id]} \\
&= \frac{\sin A}{1-\cos A} \quad \text{[Simplifying]} \\
&= \text{RHS}
\end{aligned}$$

■

28. $\frac{1+\cos A}{\sin^2 A} = \frac{1}{1-\cos A}$

$$\begin{aligned}
\text{LHS} &= \frac{1+\cos A}{\sin^2 A} \\
&= \frac{1+\cos A}{1-\cos^2 A} \quad \text{[Pythagorean Id]} \\
&= \frac{1+\cos A}{(1-\cos A)(1+\cos A)} \quad \text{[Difference of squares]} \\
&= \frac{1}{1-\cos A} \quad \text{[Simplifying]} \\
&= \text{RHS}
\end{aligned}$$

■

29. $\frac{\sin A}{1+\cos A} = \operatorname{cosec} A - \cot A$

$$\begin{aligned}
\text{LHS} &= \frac{\sin A}{1+\cos A} \\
&= \frac{\sin A(1-\cos A)}{(1+\cos A)(1-\cos A)} \quad \text{[Multiply by conjugate]} \\
&= \frac{\sin A(1-\cos A)}{1-\cos^2 A} \quad \text{[Difference of squares]} \\
&= \frac{\sin A(1-\cos A)}{\sin^2 A} \quad \text{[Pythagorean Id]} \\
&= \frac{1-\cos A}{\sin A} \quad \text{[Simplifying]} \\
&= \frac{1}{\sin A} - \frac{\cos A}{\sin A} \quad \text{[Split fraction]} \\
&= \operatorname{cosec} A - \cot A \quad \text{[Definitions]} \\
&= \text{RHS}
\end{aligned}$$

■

30. $\frac{\cos B}{1+\sin B} = \sec B - \tan B$

$$\begin{aligned}
\text{LHS} &= \frac{\cos B}{1+\sin B} \\
&= \frac{\cos B(1-\sin B)}{(1+\sin B)(1-\sin B)} \quad \text{[Multiply by conjugate]} \\
&= \frac{\cos B(1-\sin B)}{1-\sin^2 B} \quad \text{[Difference of squares]} \\
&= \frac{\cos B(1-\sin B)}{\cos^2 B} \quad \text{[Pythagorean Id]} \\
&= \frac{1-\sin B}{\cos B} \quad \text{[Simplifying]} \\
&= \frac{1}{\cos B} - \frac{\sin B}{\cos B} \quad \text{[Split fraction]} \\
&= \sec B - \tan B \quad \text{[Definitions]} \\
&= \text{RHS}
\end{aligned}$$

■

31. $\tan\theta + \cot\theta = \sec\theta \operatorname{cosec}\theta$

$$\begin{aligned}
\text{LHS} &= \tan\theta + \cot\theta \\
&= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} \quad \text{[Definitions]} \\
&= \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta} \quad \text{[Common denominator]} \\
&= \frac{1}{\cos\theta\sin\theta} \quad \text{[Pythagorean Id]} \\
&= \left(\frac{1}{\cos\theta}\right)\left(\frac{1}{\sin\theta}\right) \\
&= \sec\theta \operatorname{cosec}\theta \quad \text{[Definitions]} \\
&= \text{RHS}
\end{aligned}$$

■

32. $\cos\theta(\tan\theta + \cot\theta) = \operatorname{cosec}\theta$

$$\begin{aligned}
\text{LHS} &= \cos\theta(\tan\theta + \cot\theta) \\
&= \cos\theta(\sec\theta \operatorname{cosec}\theta) \quad \text{[Using result from 33]} \\
&= \cos\theta\left(\frac{1}{\cos\theta}\right)\operatorname{cosec}\theta \quad \text{[Definition of } \sec\theta\text{]} \\
&= \operatorname{cosec}\theta \quad \text{[Simplifying]} \\
&= \text{RHS}
\end{aligned}$$

■

33. $\sin\theta(\cot\theta+\tan\theta)=\sec\theta$

$$\begin{aligned}
\text{LHS} &= \sin\theta(\cot\theta+\tan\theta)\\
&= \sin\theta(\sec\theta\,\text{cosec}\,\theta) \quad \text{[Using result from 33]}\\
&= \sin\theta\sec\theta\left(\frac{1}{\sin\theta}\right) \quad \text{[Definition of cosec}\,\theta]\\
&= \sec\theta \quad \text{[Simplifying]}\\
&= \text{RHS}
\end{aligned}$$
■

34. $(\sin\theta+\cos\theta)^2+(\sin\theta-\cos\theta)^2=2$

$$\begin{aligned}
\text{LHS} &= (\sin\theta+\cos\theta)^2+(\sin\theta-\cos\theta)^2\\
&= (\sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta)\\
&\quad + (\sin^2\theta-2\sin\theta\cos\theta+\cos^2\theta) \quad \text{[Expanding squares]}\\
&= (\sin^2\theta+\cos^2\theta)+2\sin\theta\cos\theta\\
&\quad + (\sin^2\theta+\cos^2\theta)-2\sin\theta\cos\theta \quad \text{[Grouping]}\\
&= 1+2\sin\theta\cos\theta+1-2\sin\theta\cos\theta \quad \text{[Pythagorean Id]}\\
&= 2 \quad \text{[Simplifying]}\\
&= \text{RHS}
\end{aligned}$$
■

35. $\cos^4 A-\sin^4 A=\cos^2 A-\sin^2 A$

$$\begin{aligned}
\text{LHS} &= \cos^4 A-\sin^4 A\\
&= (\cos^2 A)^2-(\sin^2 A)^2 \quad \text{[Rewrite powers]}\\
&= (\cos^2 A-\sin^2 A)(\cos^2 A+\sin^2 A) \quad \text{[Difference of squares]}\\
&= (\cos^2 A-\sin^2 A)(1) \quad \text{[Pythagorean Id]}\\
&= \cos^2 A-\sin^2 A \quad \text{[Simplifying]}\\
&= \text{RHS}
\end{aligned}$$
■

36. $\frac{\tan x}{\sec x-1}=\frac{\sec x+1}{\tan x}$

$$\begin{aligned}
\text{LHS} &= \frac{\tan x}{\sec x-1}\\
&= \frac{\tan x(\sec x+1)}{(\sec x-1)(\sec x+1)} \quad \text{[Multiply by conjugate]}\\
&= \frac{\tan x(\sec x+1)}{\sec^2 x-1} \quad \text{[Difference of squares]}\\
&= \frac{\tan x(\sec x+1)}{\tan^2 x} \quad \text{[Pythagorean Id: } \sec^2 x-1=\tan^2 x]\\
&= \frac{\sec x+1}{\tan x} \quad \text{[Simplifying]}\\
&= \text{RHS}
\end{aligned}$$
■

37. $\frac{\cot x}{\text{cosec}\,x-1}=\frac{\text{cosec}\,x+1}{\cot x}$

$$\begin{aligned}
\text{LHS} &= \frac{\cot x}{\text{cosec}\,x-1}\\
&= \frac{\cot x(\text{cosec}\,x+1)}{(\text{cosec}\,x-1)(\text{cosec}\,x+1)} \quad \text{[Multiply by conjugate]}\\
&= \frac{\cot x(\text{cosec}\,x+1)}{\text{cosec}^2 x-1} \quad \text{[Difference of squares]}\\
&= \frac{\cot x(\text{cosec}\,x+1)}{\cot^2 x} \quad \text{[Pythagorean Id: } \text{cosec}^2 x-1=\cot^2 x]\\
&= \frac{\text{cosec}\,x+1}{\cot x} \quad \text{[Simplifying]}\\
&= \text{RHS}
\end{aligned}$$
■

38. $\frac{\sec\theta}{\operatorname{cosec}\theta} + \frac{\sin\theta}{\cos\theta} = 2\tan\theta$

$$\begin{aligned}
\text{LHS} &= \frac{\sec\theta}{\operatorname{cosec}\theta} + \frac{\sin\theta}{\cos\theta} \\
&= \frac{1/\cos\theta}{1/\sin\theta} + \tan\theta \quad \text{[Definitions]} \\
&= \frac{\sin\theta}{\cos\theta} + \tan\theta \quad \text{[Simplifying fraction]} \\
&= \tan\theta + \tan\theta \quad \text{[Definition]} \\
&= 2\tan\theta \\
&= \text{RHS}
\end{aligned}$$

■

39. $\frac{1+\sec\theta}{\sec\theta} = \frac{\sin^2\theta}{1-\cos\theta}$

$$\begin{aligned}
\text{LHS} &= \frac{1+\sec\theta}{\sec\theta} \\
&= \frac{1}{\sec\theta} + \frac{\sec\theta}{\sec\theta} \quad \text{[Split fraction]} \\
&= \cos\theta + 1 \quad \text{[Definition and simplifying]} \\
\text{RHS} &= \frac{\sin^2\theta}{1-\cos\theta} \\
&= \frac{1-\cos^2\theta}{1-\cos\theta} \quad \text{[Pythagorean Id]} \\
&= \frac{(1-\cos\theta)(1+\cos\theta)}{1-\cos\theta} \quad \text{[Difference of squares]} \\
&= 1+\cos\theta \quad \text{[Simplifying]} \\
\text{LHS} &= \text{RHS}
\end{aligned}$$

■

40. $\frac{1}{1+\sin A} + \frac{1}{1-\sin A} = 2\sec^2 A$

$$\begin{aligned}
\text{LHS} &= \frac{1}{1+\sin A} + \frac{1}{1-\sin A} \\
&= \frac{(1-\sin A)+(1+\sin A)}{(1+\sin A)(1-\sin A)} \quad \text{[Common denominator]} \\
&= \frac{2}{1-\sin^2 A} \quad \text{[Simplifying numerator]} \\
&= \frac{2}{\cos^2 A} \quad \text{[Pythagorean Id]} \\
&= 2\left(\frac{1}{\cos^2 A}\right) \\
&= 2\sec^2 A \quad \text{[Definition]} \\
&= \text{RHS}
\end{aligned}$$

■

41. $\frac{1+\sin\theta}{\cos\theta} + \frac{\cos\theta}{1+\sin\theta} = 2\sec\theta$

$$\begin{aligned}
\text{LHS} &= \frac{1+\sin\theta}{\cos\theta} + \frac{\cos\theta}{1+\sin\theta} \\
&= \frac{(1+\sin\theta)^2 + \cos^2\theta}{\cos\theta(1+\sin\theta)} \quad \text{[Common denominator]} \\
&= \frac{(1+2\sin\theta+\sin^2\theta)+\cos^2\theta}{\cos\theta(1+\sin\theta)} \quad \text{[Expand square]} \\
&= \frac{1+2\sin\theta+(\sin^2\theta+\cos^2\theta)}{\cos\theta(1+\sin\theta)} \quad \text{[Group terms]} \\
&= \frac{1+2\sin\theta+1}{\cos\theta(1+\sin\theta)} \quad \text{[Pythagorean Id]} \\
&= \frac{2+2\sin\theta}{\cos\theta(1+\sin\theta)} \quad \text{[Simplify numerator]} \\
&= \frac{2(1+\sin\theta)}{\cos\theta(1+\sin\theta)} \quad \text{[Factor numerator]} \\
&= \frac{2}{\cos\theta} \quad \text{[Simplify fraction]} \\
&= 2\sec\theta \quad \text{[Definition]} \\
&= \text{RHS}
\end{aligned}$$

■

42. $\frac{\cos\theta}{1+\sin\theta} + \frac{1+\sin\theta}{\cos\theta} = 2\sec\theta$

$$\begin{aligned}
\text{LHS} &= \frac{\cos\theta}{1+\sin\theta} + \frac{1+\sin\theta}{\cos\theta} \\
&= \frac{\cos^2\theta + (1+\sin\theta)^2}{(1+\sin\theta)\cos\theta} \quad \text{[Common denominator]} \\
&= \frac{\cos^2\theta+(1+2\sin\theta+\sin^2\theta)}{(1+\sin\theta)\cos\theta} \quad \text{[Expand square]} \\
&= \frac{(\cos^2\theta+\sin^2\theta)+1+2\sin\theta}{(1+\sin\theta)\cos\theta} \quad \text{[Group terms]} \\
&= \frac{1+1+2\sin\theta}{(1+\sin\theta)\cos\theta} \quad \text{[Pythagorean Id]} \\
&= \frac{2+2\sin\theta}{(1+\sin\theta)\cos\theta} \quad \text{[Simplify numerator]} \\
&= \frac{2(1+\sin\theta)}{(1+\sin\theta)\cos\theta} \quad \text{[Factor numerator]} \\
&= \frac{2}{\cos\theta} \quad \text{[Simplify fraction]} \\
&= 2\sec\theta \quad \text{[Definition]} \\
&= \text{RHS}
\end{aligned}$$

■

43. $\frac{1-\sin\theta}{\cos\theta} + \frac{\cos\theta}{1-\sin\theta} = 2\sec\theta$

$$\begin{aligned}
\text{LHS} &= \frac{1-\sin\theta}{\cos\theta} + \frac{\cos\theta}{1-\sin\theta} \\
&= \frac{(1-\sin\theta)^2 + \cos^2\theta}{\cos\theta(1-\sin\theta)} \quad \text{[Common denominator]} \\
&= \frac{(1-2\sin\theta+\sin^2\theta)+\cos^2\theta}{\cos\theta(1-\sin\theta)} \quad \text{[Expand square]} \\
&= \frac{1-2\sin\theta+(\sin^2\theta+\cos^2\theta)}{\cos\theta(1-\sin\theta)} \quad \text{[Group terms]} \\
&= \frac{1-2\sin\theta+1}{\cos\theta(1-\sin\theta)} \quad \text{[Pythagorean Id]} \\
&= \frac{2-2\sin\theta}{\cos\theta(1-\sin\theta)} \quad \text{[Simplify numerator]} \\
&= \frac{2(1-\sin\theta)}{\cos\theta(1-\sin\theta)} \quad \text{[Factor numerator]} \\
&= \frac{2}{\cos\theta} \quad \text{[Simplify fraction]} \\
&= 2\sec\theta \quad \text{[Definition]} \\
&= \text{RHS}
\end{aligned}$$

■

44. $\frac{1}{1-\sin x} - \frac{1}{1+\sin x} = 2\tan x\sec x$

$$\begin{aligned}
\text{LHS} &= \frac{1}{1-\sin x} - \frac{1}{1+\sin x} \\
&= \frac{(1+\sin x)-(1-\sin x)}{(1-\sin x)(1+\sin x)} \quad \text{[Common denominator]} \\
&= \frac{1+\sin x-1+\sin x}{1-\sin^2 x} \quad \text{[Simplify numerator, Diff of squares]} \\
&= \frac{2\sin x}{\cos^2 x} \quad \text{[Simplify numerator, Pythagorean Id]} \\
&= 2\left(\frac{\sin x}{\cos x}\right)\left(\frac{1}{\cos x}\right) \quad \text{[Split factors]} \\
&= 2\tan x\sec x \quad \text{[Definitions]} \\
&= \text{RHS}
\end{aligned}$$

■

45. $\frac{1}{1-\cos x} - \frac{1}{1+\cos x} = 2\cot x\,\text{cosec}\,x$

$$\begin{aligned}
\text{LHS} &= \frac{1}{1-\cos x} - \frac{1}{1+\cos x} \\
&= \frac{(1+\cos x)-(1-\cos x)}{(1-\cos x)(1+\cos x)} \quad \text{[Common denominator]} \\
&= \frac{1+\cos x-1+\cos x}{1-\cos^2 x} \quad \text{[Simplify numerator, Diff of squares]} \\
&= \frac{2\cos x}{\sin^2 x} \quad \text{[Simplify numerator, Pythagorean Id]} \\
&= 2\left(\frac{\cos x}{\sin x}\right)\left(\frac{1}{\sin x}\right) \quad \text{[Split factors]} \\
&= 2\cot x\,\text{cosec}\,x \quad \text{[Definitions]} \\
&= \text{RHS}
\end{aligned}$$

■

46. $\frac{1}{\sec A-1} + \frac{1}{\sec A+1} = 2\operatorname{cosec} A \cot A$

$$
\begin{aligned}
\text{LHS} &= \frac{1}{\sec A - 1} + \frac{1}{\sec A + 1} \\
&= \frac{(\sec A + 1) + (\sec A - 1)}{(\sec A - 1)(\sec A + 1)} \quad \text{[Common denominator]} \\
&= \frac{2\sec A}{\sec^2 A - 1} \quad \text{[Simplify numerator, Diff of squares]} \\
&= \frac{2\sec A}{\tan^2 A} \quad \text{[Pythagorean Id]} \\
&= \frac{2(1/\cos A)}{\sin^2 A/\cos^2 A} \quad \text{[Definitions]} \\
&= \frac{2}{\cos A} \cdot \frac{\cos^2 A}{\sin^2 A} \quad \text{[Fraction division]} \\
&= \frac{2\cos A}{\sin^2 A} \quad \text{[Simplifying]} \\
&= 2\left(\frac{1}{\sin A}\right)\left(\frac{\cos A}{\sin A}\right) \quad \text{[Split factors]} \\
&= 2\operatorname{cosec} A \cot A \quad \text{[Definitions]} \\
&= \text{RHS}
\end{aligned}
$$

■

47. $\frac{\sin\alpha}{1+\cos\alpha} + \frac{1+\cos\alpha}{\sin\alpha} = 2\operatorname{cosec}\alpha$

$$
\begin{aligned}
\text{LHS} &= \frac{\sin\alpha}{1 + \cos\alpha} + \frac{1 + \cos\alpha}{\sin\alpha} \\
&= \frac{\sin^2\alpha + (1 + \cos\alpha)^2}{\sin\alpha(1 + \cos\alpha)} \quad \text{[Common denominator]} \\
&= \frac{\sin^2\alpha + (1 + 2\cos\alpha + \cos^2\alpha)}{\sin\alpha(1 + \cos\alpha)} \quad \text{[Expand square]} \\
&= \frac{(\sin^2\alpha + \cos^2\alpha) + 1 + 2\cos\alpha}{\sin\alpha(1 + \cos\alpha)} \quad \text{[Group terms]} \\
&= \frac{1 + 1 + 2\cos\alpha}{\sin\alpha(1 + \cos\alpha)} \quad \text{[Pythagorean Id]} \\
&= \frac{2 + 2\cos\alpha}{\sin\alpha(1 + \cos\alpha)} \quad \text{[Simplify numerator]} \\
&= \frac{2(1 + \cos\alpha)}{\sin\alpha(1 + \cos\alpha)} \quad \text{[Factor numerator]} \\
&= \frac{2}{\sin\alpha} \quad \text{[Simplify fraction]} \\
&= 2\operatorname{cosec}\alpha \quad \text{[Definition]} \\
&= \text{RHS}
\end{aligned}
$$

■

48. $\frac{\tan\beta}{1+\sec\beta} + \frac{1+\sec\beta}{\tan\beta} = 2\operatorname{cosec}\beta$

$$\begin{aligned}
\text{LHS} &= \frac{\tan\beta}{1+\sec\beta} + \frac{1+\sec\beta}{\tan\beta} \\
&= \frac{\tan^2\beta + (1+\sec\beta)^2}{\tan\beta(1+\sec\beta)} \quad \text{[Common denominator]} \\
&= \frac{\tan^2\beta + (1+2\sec\beta+\sec^2\beta)}{\tan\beta(1+\sec\beta)} \quad \text{[Expand square]} \\
&= \frac{(\tan^2\beta+\sec^2\beta)+1+2\sec\beta}{\tan\beta(1+\sec\beta)} \\
&= \frac{(\sec^2\beta-1+\sec^2\beta)+1+2\sec\beta}{\tan\beta(1+\sec\beta)} \quad \text{[Pythagorean Id]} \\
&= \frac{2\sec^2\beta+2\sec\beta}{\tan\beta(1+\sec\beta)} \quad \text{[Simplify numerator]} \\
&= \frac{2\sec\beta(\sec\beta+1)}{\tan\beta(1+\sec\beta)} \quad \text{[Factor numerator]} \\
&= \frac{2\sec\beta}{\tan\beta} \quad \text{[Simplify fraction]} \\
&= \frac{2(1/\cos\beta)}{\sin\beta/\cos\beta} \quad \text{[Definitions]} \\
&= \frac{2}{\cos\beta}\cdot\frac{\cos\beta}{\sin\beta} \\
&= \frac{2}{\sin\beta} \quad \text{[Simplifying]} \\
&= 2\operatorname{cosec}\beta \quad \text{[Definition]} \\
&= \text{RHS}
\end{aligned}$$

■

49. $\frac{\cot\theta}{1+\operatorname{cosec}\theta} - \frac{\cot\theta}{1-\operatorname{cosec}\theta} = 2\sec\theta$

$$\begin{aligned}
\text{LHS} &= \cot\theta\left(\frac{1}{1+\operatorname{cosec}\theta} - \frac{1}{1-\operatorname{cosec}\theta}\right) \\
&= \cot\theta\left(\frac{(1-\operatorname{cosec}\theta)-(1+\operatorname{cosec}\theta)}{(1+\operatorname{cosec}\theta)(1-\operatorname{cosec}\theta)}\right) \quad \text{[Common denominator]} \\
&= \cot\theta\left(\frac{1-\operatorname{cosec}\theta-1-\operatorname{cosec}\theta}{1-\operatorname{cosec}^2\theta}\right) \quad \text{[Simplify num, Diff squares]} \\
&= \cot\theta\left(\frac{-2\operatorname{cosec}\theta}{-\cot^2\theta}\right) \quad \text{[Simplify num, Pythagorean Id]} \\
&= \cot\theta\left(\frac{2\operatorname{cosec}\theta}{\cot^2\theta}\right) \quad \text{[Simplify signs]} \\
&= \frac{2\operatorname{cosec}\theta}{\cot\theta} \quad \text{[Simplify fraction]} \\
&= \frac{2(1/\sin\theta)}{\cos\theta/\sin\theta} \quad \text{[Definitions]} \\
&= \frac{2}{\sin\theta}\cdot\frac{\sin\theta}{\cos\theta} \\
&= \frac{2}{\cos\theta} \quad \text{[Simplifying]} \\
&= 2\sec\theta \quad \text{[Definition]} \\
&= \text{RHS}
\end{aligned}$$

■

50. $\frac{\tan\theta}{1+\sec\theta} - \frac{\tan\theta}{1-\sec\theta} = 2\operatorname{cosec}\theta$

$$\begin{aligned}
\text{LHS} &= \tan\theta\left(\frac{1}{1+\sec\theta} - \frac{1}{1-\sec\theta}\right) \\
&= \tan\theta\left(\frac{(1-\sec\theta)-(1+\sec\theta)}{(1+\sec\theta)(1-\sec\theta)}\right) \quad \text{[Common denominator]} \\
&= \tan\theta\left(\frac{1-\sec\theta-1-\sec\theta}{1-\sec^2\theta}\right) \quad \text{[Simplify num, Diff squares]} \\
&= \tan\theta\left(\frac{-2\sec\theta}{-\tan^2\theta}\right) \quad \text{[Simplify num, Pythagorean Id]} \\
&= \tan\theta\left(\frac{2\sec\theta}{\tan^2\theta}\right) \quad \text{[Simplify signs]} \\
&= \frac{2\sec\theta}{\tan\theta} \quad \text{[Simplify fraction]} \\
&= \frac{2(1/\cos\theta)}{\sin\theta/\cos\theta} \quad \text{[Definitions]} \\
&= \frac{2}{\cos\theta}\cdot\frac{\cos\theta}{\sin\theta} \\
&= \frac{2}{\sin\theta} \quad \text{[Simplifying]} \\
&= 2\operatorname{cosec}\theta \quad \text{[Definition]} \\
&= \text{RHS}
\end{aligned}$$

■

51. $\frac{\operatorname{cosec}A}{\operatorname{cosec}A-1} + \frac{\operatorname{cosec}A}{\operatorname{cosec}A+1} = 2\sec^2 A$

$$\begin{aligned}
\text{LHS} &= \operatorname{cosec}A\left(\frac{1}{\operatorname{cosec}A-1} + \frac{1}{\operatorname{cosec}A+1}\right) \\
&= \operatorname{cosec}A\left(\frac{(\operatorname{cosec}A+1)+(\operatorname{cosec}A-1)}{(\operatorname{cosec}A-1)(\operatorname{cosec}A+1)}\right) \quad \text{[Common denominator]} \\
&= \operatorname{cosec}A\left(\frac{2\operatorname{cosec}A}{\operatorname{cosec}^2A-1}\right) \quad \text{[Simplify num, Diff squares]} \\
&= \operatorname{cosec}A\left(\frac{2\operatorname{cosec}A}{\cot^2A}\right) \quad \text{[Pythagorean Id]} \\
&= \frac{2\operatorname{cosec}^2A}{\cot^2A} \quad \text{[Multiply]} \\
&= \frac{2(1/\sin^2A)}{\cos^2A/\sin^2A} \quad \text{[Definitions]} \\
&= \frac{2}{\sin^2A}\cdot\frac{\sin^2A}{\cos^2A} \\
&= \frac{2}{\cos^2A} \quad \text{[Simplifying]} \\
&= 2\sec^2A \quad \text{[Definition]} \\
&= \text{RHS}
\end{aligned}$$

■

52. $(\sec^2\theta - 1)(\operatorname{cosec}^2\theta - 1) = 1$

$$\begin{aligned}
\text{LHS (corrected)} &= (\sec^2\theta-1)(\operatorname{cosec}^2\theta-1) \\
&= (\tan^2\theta)(\cot^2\theta) \quad \text{[Pythagorean Ids]} \\
&= \tan^2\theta\left(\frac{1}{\tan^2\theta}\right) \quad \text{[Definition]} \\
&= 1 \\
&= \text{RHS}
\end{aligned}$$

53. $(\sec A - \tan A)^2 = \frac{1-\sin A}{1+\sin A}$

$$\begin{aligned}
\text{LHS} &= (\sec A - \tan A)^2 \\
&= \left(\frac{1}{\cos A} - \frac{\sin A}{\cos A}\right)^2 \quad \text{[Definitions]} \\
&= \left(\frac{1-\sin A}{\cos A}\right)^2 \\
&= \frac{(1-\sin A)^2}{\cos^2 A} \quad \text{[Power rule]} \\
&= \frac{(1-\sin A)^2}{1-\sin^2 A} \quad \text{[Pythagorean Id]} \\
&= \frac{(1-\sin A)^2}{(1-\sin A)(1+\sin A)} \quad \text{[Difference of squares]} \\
&= \frac{1-\sin A}{1+\sin A} \quad \text{[Simplifying]} \\
&= \text{RHS}
\end{aligned}$$

■

54. $\frac{1-\sin\theta}{1+\sin\theta} = (\sec\theta - \tan\theta)^2$

$$\begin{aligned}
\text{RHS} &= (\sec\theta - \tan\theta)^2 \\
&= \left(\frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}\right)^2 \quad \text{[Definitions]} \\
&= \left(\frac{1-\sin\theta}{\cos\theta}\right)^2 \\
&= \frac{(1-\sin\theta)^2}{\cos^2\theta} \quad \text{[Power rule]} \\
&= \frac{(1-\sin\theta)^2}{1-\sin^2\theta} \quad \text{[Pythagorean Id]} \\
&= \frac{(1-\sin\theta)^2}{(1-\sin\theta)(1+\sin\theta)} \quad \text{[Difference of squares]} \\
&= \frac{1-\sin\theta}{1+\sin\theta} \quad \text{[Simplifying]} \\
&= \text{LHS}
\end{aligned}$$

■

55. $\frac{1-\cos A}{1+\cos A} = (\operatorname{cosec} A - \cot A)^2$

$$\begin{aligned}
\text{RHS} &= (\operatorname{cosec} A - \cot A)^2 \\
&= \left(\frac{1}{\sin A} - \frac{\cos A}{\sin A}\right)^2 \quad \text{[Definitions]} \\
&= \left(\frac{1-\cos A}{\sin A}\right)^2 \\
&= \frac{(1-\cos A)^2}{\sin^2 A} \quad \text{[Power rule]} \\
&= \frac{(1-\cos A)^2}{1-\cos^2 A} \quad \text{[Pythagorean Id]} \\
&= \frac{(1-\cos A)^2}{(1-\cos A)(1+\cos A)} \quad \text{[Difference of squares]} \\
&= \frac{1-\cos A}{1+\cos A} \quad \text{[Simplifying]} \\
&= \text{LHS}
\end{aligned}$$

■

56. $\operatorname{cosec}\theta\sqrt{1-\cos^2\theta} = 1$ (Assume $\sin\theta > 0$)

$$\begin{aligned}
\text{LHS} &= \operatorname{cosec}\theta\sqrt{1-\cos^2\theta} \\
&= \operatorname{cosec}\theta\sqrt{\sin^2\theta} \quad \text{[Pythagorean Id]} \\
&= \operatorname{cosec}\theta|\sin\theta| \quad \text{[Definition of square root]} \\
&= \operatorname{cosec}\theta\sin\theta \quad \text{[Since } \sin\theta > 0\text{]} \\
&= \left(\frac{1}{\sin\theta}\right)\sin\theta \quad \text{[Definition]} \\
&= 1 \quad \text{[Simplifying]} \\
&= \text{RHS}
\end{aligned}$$

■

57. $\sec^2 x + \operatorname{cosec}^2 x = \sec^2 x\operatorname{cosec}^2 x$

$$\begin{aligned}
\text{LHS} &= \sec^2 x + \operatorname{cosec}^2 x \\
&= \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} \quad \text{[Definitions]} \\
&= \frac{\sin^2 x + \cos^2 x}{\cos^2 x\sin^2 x} \quad \text{[Common denominator]} \\
&= \frac{1}{\cos^2 x\sin^2 x} \quad \text{[Pythagorean Id]} \\
&= \left(\frac{1}{\cos^2 x}\right)\left(\frac{1}{\sin^2 x}\right) \\
&= \sec^2 x\operatorname{cosec}^2 x \quad \text{[Definitions]} \\
&= \text{RHS}
\end{aligned}$$

■

58. $(1+\tan^2\theta)(1-\sin\theta)(1+\sin\theta) = 1$

$$\begin{aligned}
\text{LHS} &= (1+\tan^2\theta)(1-\sin\theta)(1+\sin\theta) \\
&= (\sec^2\theta)(1-\sin^2\theta) \quad \text{[Pythagorean Id, Diff of squares]} \\
&= (\sec^2\theta)(\cos^2\theta) \quad \text{[Pythagorean Id]} \\
&= \left(\frac{1}{\cos^2\theta}\right)\cos^2\theta \quad \text{[Definition]} \\
&= 1 \quad \text{[Simplifying]} \\
&= \text{RHS}
\end{aligned}$$

■

59. $\sin^2 A\cot^2 A + \cos^2 A\tan^2 A = 1$

$$\begin{aligned}
\text{LHS} &= \sin^2 A\cot^2 A + \cos^2 A\tan^2 A \\
&= \sin^2 A\left(\frac{\cos^2 A}{\sin^2 A}\right) + \cos^2 A\left(\frac{\sin^2 A}{\cos^2 A}\right) \quad \text{[Definitions]} \\
&= \cos^2 A + \sin^2 A \quad \text{[Simplifying]} \\
&= 1 \quad \text{[Pythagorean Id]} \\
&= \text{RHS}
\end{aligned}$$

■

60. $\tan^2\theta - \sin^2\theta = \tan^2\theta\sin^2\theta$

$$\begin{aligned}
\text{LHS} &= \tan^2\theta - \sin^2\theta \\
&= \frac{\sin^2\theta}{\cos^2\theta} - \sin^2\theta \quad \text{[Definition]} \\
&= \sin^2\theta\left(\frac{1}{\cos^2\theta} - 1\right) \quad \text{[Factor out } \sin^2\theta\text{]} \\
&= \sin^2\theta\left(\sec^2\theta - 1\right) \quad \text{[Definition]} \\
&= \sin^2\theta(\tan^2\theta) \quad \text{[Pythagorean Id]} \\
&= \tan^2\theta\sin^2\theta \\
&= \text{RHS}
\end{aligned}$$

■

61. $\cot\theta - \tan\theta = \frac{2\cos^2\theta - 1}{\sin\theta\cos\theta}$

$$\begin{aligned}
\text{LHS} &= \cot\theta - \tan\theta \\
&= \frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta} \quad \text{[Definitions]} \\
&= \frac{\cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta} \quad \text{[Common denominator]} \\
&= \frac{\cos^2\theta - (1 - \cos^2\theta)}{\sin\theta\cos\theta} \quad \text{[Pythagorean Id]} \\
&= \frac{\cos^2\theta - 1 + \cos^2\theta}{\sin\theta\cos\theta} \\
&= \frac{2\cos^2\theta - 1}{\sin\theta\cos\theta} \quad \text{[Simplify numerator]} \\
&= \text{RHS}
\end{aligned}$$

■

62. $\tan^2 x - \cot^2 x = \sec^2 x - \operatorname{cosec}^2 x$

$$\begin{aligned}
\text{LHS} &= \tan^2 x - \cot^2 x \\
&= (\sec^2 x - 1) - (\operatorname{cosec}^2 x - 1) \quad \text{[Pythagorean Ids]} \\
&= \sec^2 x - 1 - \operatorname{cosec}^2 x + 1 \\
&= \sec^2 x - \operatorname{cosec}^2 x \quad \text{[Simplifying]} \\
&= \text{RHS}
\end{aligned}$$

■

63. $(\operatorname{cosec}\theta + \sin\theta)(\operatorname{cosec}\theta - \sin\theta) = \cot^2\theta + \cos^2\theta$

$$\begin{aligned}
\text{LHS} &= (\operatorname{cosec}\theta + \sin\theta)(\operatorname{cosec}\theta - \sin\theta) \\
&= \operatorname{cosec}^2\theta - \sin^2\theta \quad \text{[Difference of squares]} \\
&= (1 + \cot^2\theta) - (1 - \cos^2\theta) \quad \text{[Pythagorean Ids]} \\
&= 1 + \cot^2\theta - 1 + \cos^2\theta \\
&= \cot^2\theta + \cos^2\theta \quad \text{[Simplifying]} \\
&= \text{RHS}
\end{aligned}$$

■

64. $(\sec\theta + \cos\theta)(\sec\theta - \cos\theta) = \tan^2\theta + \sin^2\theta$

$$\begin{aligned}
\text{LHS} &= (\sec\theta + \cos\theta)(\sec\theta - \cos\theta) \\
&= \sec^2\theta - \cos^2\theta \quad \text{[Difference of squares]} \\
&= (1 + \tan^2\theta) - (1 - \sin^2\theta) \quad \text{[Pythagorean Ids]} \\
&= 1 + \tan^2\theta - 1 + \sin^2\theta \\
&= \tan^2\theta + \sin^2\theta \quad \text{[Simplifying]} \\
&= \text{RHS}
\end{aligned}$$

■

65. $\sec A(1 - \sin A)(\sec A + \tan A) = 1$

$$\begin{aligned}
\text{LHS} &= \sec A(1 - \sin A)(\sec A + \tan A) \\
&= (\sec A - \sec A\sin A)(\sec A + \tan A) \quad \text{[Distribute]} \\
&= \left(\sec A - \frac{1}{\cos A}\sin A\right)(\sec A + \tan A) \quad \text{[Definition]} \\
&= (\sec A - \tan A)(\sec A + \tan A) \quad \text{[Definition]} \\
&= \sec^2 A - \tan^2 A \quad \text{[Difference of squares]} \\
&= (1 + \tan^2 A) - \tan^2 A \quad \text{[Pythagorean Id]} \\
&= 1 \quad \text{[Simplifying]} \\
&= \text{RHS}
\end{aligned}$$

■

66. $\frac{\cos^2\theta}{\sin\theta} - \operatorname{cosec}\theta + \sin\theta = 0$

$$\begin{aligned}
\text{LHS} &= \frac{\cos^2\theta}{\sin\theta} - \operatorname{cosec}\theta + \sin\theta \\
&= \frac{\cos^2\theta}{\sin\theta} - \frac{1}{\sin\theta} + \frac{\sin^2\theta}{\sin\theta} \quad \text{[Def, Common denominator]} \\
&= \frac{\cos^2\theta - 1 + \sin^2\theta}{\sin\theta} \\
&= \frac{(\cos^2\theta + \sin^2\theta) - 1}{\sin\theta} \quad \text{[Group terms]} \\
&= \frac{1-1}{\sin\theta} \quad \text{[Pythagorean Id]} \\
&= \frac{0}{\sin\theta} \\
&= 0 \quad [\text{Assuming } \sin\theta \neq 0] \\
&= \text{RHS}
\end{aligned}$$

■

67. $\frac{(\sin\theta+\cos\theta)^2}{\cos\theta} - \sec\theta = 2\sin\theta$

$$\begin{aligned}
\text{LHS} &= \frac{(\sin\theta+\cos\theta)^2}{\cos\theta} - \sec\theta \\
&= \frac{\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta}{\cos\theta} - \frac{1}{\cos\theta} \quad \text{[Expand, Definition]} \\
&= \frac{(\sin^2\theta + \cos^2\theta) + 2\sin\theta\cos\theta}{\cos\theta} - \frac{1}{\cos\theta} \\
&= \frac{1 + 2\sin\theta\cos\theta}{\cos\theta} - \frac{1}{\cos\theta} \quad \text{[Pythagorean Id]} \\
&= \frac{1 + 2\sin\theta\cos\theta - 1}{\cos\theta} \quad \text{[Combine fractions]} \\
&= \frac{2\sin\theta\cos\theta}{\cos\theta} \\
&= 2\sin\theta \quad \text{[Simplifying]} \\
&= \text{RHS}
\end{aligned}$$

■

68. $\frac{1+\cos\theta-\sin^2\theta}{\sin\theta(1+\cos\theta)} = \cot\theta$

$$\begin{aligned}
\text{LHS} &= \frac{1+\cos\theta-\sin^2\theta}{\sin\theta(1+\cos\theta)} \\
&= \frac{(1-\sin^2\theta)+\cos\theta}{\sin\theta(1+\cos\theta)} \quad \text{[Group terms]} \\
&= \frac{\cos^2\theta+\cos\theta}{\sin\theta(1+\cos\theta)} \quad \text{[Pythagorean Id]} \\
&= \frac{\cos\theta(\cos\theta+1)}{\sin\theta(1+\cos\theta)} \quad \text{[Factor numerator]} \\
&= \frac{\cos\theta}{\sin\theta} \quad \text{[Simplifying]} \\
&= \cot\theta \quad \text{[Definition]} \\
&= \text{RHS}
\end{aligned}$$

■

69. $\sec^4\theta - \sec^2\theta = \tan^4\theta + \tan^2\theta$

$$\begin{aligned}
\text{LHS} &= \sec^4\theta - \sec^2\theta \\
&= \sec^2\theta(\sec^2\theta - 1) \quad \text{[Factor]} \\
&= (1+\tan^2\theta)(\tan^2\theta) \quad \text{[Pythagorean Ids]} \\
&= \tan^2\theta + \tan^4\theta \quad \text{[Distribute]} \\
&= \tan^4\theta + \tan^2\theta \\
&= \text{RHS}
\end{aligned}$$

■

70. $\operatorname{cosec}^4\theta - \operatorname{cosec}^2\theta = \cot^4\theta + \cot^2\theta$

$$\begin{aligned}
\text{LHS} &= \operatorname{cosec}^4\theta - \operatorname{cosec}^2\theta \\
&= \operatorname{cosec}^2\theta(\operatorname{cosec}^2\theta - 1) \quad \text{[Factor]} \\
&= (1+\cot^2\theta)(\cot^2\theta) \quad \text{[Pythagorean Ids]} \\
&= \cot^2\theta + \cot^4\theta \quad \text{[Distribute]} \\
&= \cot^4\theta + \cot^2\theta \\
&= \text{RHS}
\end{aligned}$$

■

71. $\sec^4 A - \tan^4 A = \sec^2 A + \tan^2 A$

$$\begin{aligned}
\text{LHS} &= \sec^4 A - \tan^4 A \\
&= (\sec^2 A)^2 - (\tan^2 A)^2 \\
&= (\sec^2 A - \tan^2 A)(\sec^2 A + \tan^2 A) \quad \text{[Difference of squares]} \\
&= ((1+\tan^2 A) - \tan^2 A)(\sec^2 A + \tan^2 A) \quad \text{[Pythagorean Id]} \\
&= (1)(\sec^2 A + \tan^2 A) \\
&= \sec^2 A + \tan^2 A \\
&= \text{RHS}
\end{aligned}$$

■

72. $\operatorname{cosec}^4 A - \cot^4 A = \operatorname{cosec}^2 A + \cot^2 A$

$$\begin{aligned}
\text{LHS} &= \operatorname{cosec}^4 A - \cot^4 A \\
&= (\operatorname{cosec}^2 A)^2 - (\cot^2 A)^2 \\
&= (\operatorname{cosec}^2 A - \cot^2 A)(\operatorname{cosec}^2 A + \cot^2 A) \quad \text{[Difference of squares]} \\
&= ((1+\cot^2 A) - \cot^2 A)(\operatorname{cosec}^2 A + \cot^2 A) \quad \text{[Pythagorean Id]} \\
&= (1)(\operatorname{cosec}^2 A + \cot^2 A) \\
&= \operatorname{cosec}^2 A + \cot^2 A \\
&= \text{RHS}
\end{aligned}$$

■

73. $\sin^4\theta + \cos^4\theta = 1 - 2\sin^2\theta\cos^2\theta$

$$\begin{aligned}
\text{LHS} &= \sin^4\theta + \cos^4\theta \\
&= (\sin^2\theta)^2 + (\cos^2\theta)^2 \\
&= (\sin^2\theta + \cos^2\theta)^2 - 2(\sin^2\theta)(\cos^2\theta) \quad \text{[Using } a^2+b^2 = (a+b)^2 - 2ab\text{]} \\
&= (1)^2 - 2\sin^2\theta\cos^2\theta \quad \text{[Pythagorean Id]} \\
&= 1 - 2\sin^2\theta\cos^2\theta \\
&= \text{RHS}
\end{aligned}$$

■

74. $\frac{\tan\theta-\cot\theta}{\tan\theta+\cot\theta} = \sin^2\theta - \cos^2\theta$

$$\begin{aligned}
\text{LHS} &= \frac{\tan\theta - \cot\theta}{\tan\theta + \cot\theta} \\
&= \frac{\frac{\sin\theta}{\cos\theta} - \frac{\cos\theta}{\sin\theta}}{\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}} \quad \text{[Definitions]} \\
&= \frac{\frac{\sin^2\theta-\cos^2\theta}{\cos\theta\sin\theta}}{\frac{\sin^2\theta+\cos^2\theta}{\cos\theta\sin\theta}} \quad \text{[Common denominators]} \\
&= \frac{\sin^2\theta - \cos^2\theta}{\sin^2\theta + \cos^2\theta} \quad \text{[Simplify fraction]} \\
&= \frac{\sin^2\theta - \cos^2\theta}{1} \quad \text{[Pythagorean Id]} \\
&= \sin^2\theta - \cos^2\theta \\
&= \text{RHS}
\end{aligned}$$

■

75. $\frac{\cot A-\cos A}{\cot A+\cos A} = \frac{\operatorname{cosec} A-1}{\operatorname{cosec} A+1}$

$$\begin{aligned}
\text{LHS} &= \frac{\cot A - \cos A}{\cot A + \cos A} \\
&= \frac{\frac{\cos A}{\sin A} - \cos A}{\frac{\cos A}{\sin A} + \cos A} \quad \text{[Definition]} \\
&= \frac{\cos A\left(\frac{1}{\sin A} - 1\right)}{\cos A\left(\frac{1}{\sin A} + 1\right)} \quad \text{[Factor out } \cos A\text{]} \\
&= \frac{\frac{1}{\sin A} - 1}{\frac{1}{\sin A} + 1} \quad \text{[Simplify fraction]} \\
&= \frac{\operatorname{cosec} A - 1}{\operatorname{cosec} A + 1} \quad \text{[Definition]} \\
&= \text{RHS}
\end{aligned}$$

■

76. $\frac{\sec\theta+\tan\theta}{\cot\theta+\cos\theta} = \tan\theta\sec\theta$

$$\begin{aligned}
\text{LHS} &= \frac{\sec\theta + \tan\theta}{\cot\theta + \cos\theta} \\
&= \frac{\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}}{\frac{\cos\theta}{\sin\theta} + \cos\theta} \quad \text{[Definitions]} \\
&= \frac{\frac{1+\sin\theta}{\cos\theta}}{\frac{\cos\theta+\cos\theta\sin\theta}{\sin\theta}} \quad \text{[Common denominators]} \\
&= \frac{\frac{1+\sin\theta}{\cos\theta}}{\frac{\cos\theta(1+\sin\theta)}{\sin\theta}} \quad \text{[Factor denominator]} \\
&= \frac{1+\sin\theta}{\cos\theta} \cdot \frac{\sin\theta}{\cos\theta(1+\sin\theta)} \quad \text{[Fraction division]} \\
&= \frac{\sin\theta}{\cos^2\theta} \quad \text{[Simplifying]} \\
&= \left(\frac{\sin\theta}{\cos\theta}\right)\left(\frac{1}{\cos\theta}\right) \\
&= \tan\theta\sec\theta \quad \text{[Definitions]} \\
&= \text{RHS}
\end{aligned}$$

■

77. $\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = |\operatorname{cosec}\theta - \cot\theta|$

$$\begin{aligned}
\text{LHS} &= \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \\
&= \sqrt{\frac{(1-\cos\theta)(1-\cos\theta)}{(1+\cos\theta)(1-\cos\theta)}} \quad \text{[Multiply by conjugate inside root]} \\
&= \sqrt{\frac{(1-\cos\theta)^2}{1-\cos^2\theta}} \quad \text{[Simplify num, Diff squares]} \\
&= \sqrt{\frac{(1-\cos\theta)^2}{\sin^2\theta}} \quad \text{[Pythagorean Id]} \\
&= \sqrt{\left(\frac{1-\cos\theta}{\sin\theta}\right)^2} \\
&= \left|\frac{1-\cos\theta}{\sin\theta}\right| \quad \text{[Definition of square root]} \\
&= \left|\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\right| \quad \text{[Split fraction]} \\
&= |\operatorname{cosec}\theta - \cot\theta| \quad \text{[Definitions]} \\
&= \text{RHS}
\end{aligned}$$

■

78. $\sqrt{\frac{1+\sin A}{1-\sin A}} = |\sec A + \tan A|$

$$\begin{aligned}
\text{LHS} &= \sqrt{\frac{1+\sin A}{1-\sin A}} \\
&= \sqrt{\frac{(1+\sin A)(1+\sin A)}{(1-\sin A)(1+\sin A)}} \quad \text{[Multiply by conjugate inside root]} \\
&= \sqrt{\frac{(1+\sin A)^2}{1-\sin^2 A}} \quad \text{[Simplify num, Diff squares]} \\
&= \sqrt{\frac{(1+\sin A)^2}{\cos^2 A}} \quad \text{[Pythagorean Id]} \\
&= \sqrt{\left(\frac{1+\sin A}{\cos A}\right)^2} \\
&= \left|\frac{1+\sin A}{\cos A}\right| \quad \text{[Definition of square root]} \\
&= \left|\frac{1}{\cos A} + \frac{\sin A}{\cos A}\right| \quad \text{[Split fraction]} \\
&= |\sec A + \tan A| \quad \text{[Definitions]} \\
&= \text{RHS}
\end{aligned}$$

■

79. $\sqrt{\frac{1-\cos A}{1+\cos A}} = |\operatorname{cosec} A - \cot A|$ *Note: This is identical to question 79.*

80. $\sqrt{\sec^2\theta + \operatorname{cosec}^2\theta} = |\tan\theta + \cot\theta|$

$$\begin{aligned}
\text{LHS} &= \sqrt{\sec^2\theta + \operatorname{cosec}^2\theta} \\
&= \sqrt{\frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta}} \quad \text{[Definitions]} \\
&= \sqrt{\frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta\sin^2\theta}} \quad \text{[Common denominator]} \\
&= \sqrt{\frac{1}{\cos^2\theta\sin^2\theta}} \quad \text{[Pythagorean Id]} \\
&= \sqrt{\left(\frac{1}{\cos\theta\sin\theta}\right)^2} \\
&= \left|\frac{1}{\cos\theta\sin\theta}\right| \quad \text{[Definition of square root]} \\
\text{RHS} &= |\tan\theta + \cot\theta| \\
&= \left|\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right| \quad \text{[Definitions]} \\
&= \left|\frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}\right| \quad \text{[Common denominator]} \\
&= \left|\frac{1}{\cos\theta\sin\theta}\right| \quad \text{[Pythagorean Id]} \\
\text{LHS} &= \text{RHS}
\end{aligned}$$

■

81. $\frac{\tan^3\theta - 1}{\tan\theta - 1} = \sec^2\theta + \tan\theta$

$$\begin{aligned}
\text{LHS} &= \frac{\tan^3\theta - 1}{\tan\theta - 1} \\
&= \frac{(\tan\theta - 1)(\tan^2\theta + \tan\theta + 1)}{\tan\theta - 1} \quad \text{[Difference of cubes]} \\
&= \tan^2\theta + \tan\theta + 1 \quad \text{[Simplifying]} \\
&= (1 + \tan^2\theta) + \tan\theta \quad \text{[Group terms]} \\
&= \sec^2\theta + \tan\theta \quad \text{[Pythagorean Id]} \\
&= \text{RHS}
\end{aligned}$$

■

82. $\frac{\cot^3\theta-1}{\cot\theta-1} = \operatorname{cosec}^2\theta + \cot\theta$

$$\begin{aligned}
\text{LHS} &= \frac{\cot^3\theta - 1}{\cot\theta - 1} \\
&= \frac{(\cot\theta - 1)(\cot^2\theta + \cot\theta + 1)}{\cot\theta - 1} \quad \text{[Difference of cubes]} \\
&= \cot^2\theta + \cot\theta + 1 \quad \text{[Simplifying]} \\
&= (1 + \cot^2\theta) + \cot\theta \quad \text{[Group terms]} \\
&= \operatorname{cosec}^2\theta + \cot\theta \quad \text{[Pythagorean Id]} \\
&= \text{RHS}
\end{aligned}$$

■

83. $\frac{\sin^3 x+\cos^3 x}{\sin x+\cos x} = 1 - \sin x\cos x$

$$\begin{aligned}
\text{LHS} &= \frac{\sin^3 x + \cos^3 x}{\sin x + \cos x} \\
&= \frac{(\sin x + \cos x)(\sin^2 x - \sin x\cos x + \cos^2 x)}{\sin x + \cos x} \quad \text{[Sum of cubes]} \\
&= \sin^2 x - \sin x\cos x + \cos^2 x \quad \text{[Simplifying]} \\
&= (\sin^2 x + \cos^2 x) - \sin x\cos x \quad \text{[Group terms]} \\
&= 1 - \sin x\cos x \quad \text{[Pythagorean Id]} \\
&= \text{RHS}
\end{aligned}$$

■

84. $\frac{\sin^3 x-\cos^3 x}{\sin x-\cos x} = 1 + \sin x\cos x$

$$\begin{aligned}
\text{LHS} &= \frac{\sin^3 x - \cos^3 x}{\sin x - \cos x} \\
&= \frac{(\sin x - \cos x)(\sin^2 x + \sin x\cos x + \cos^2 x)}{\sin x - \cos x} \quad \text{[Difference of cubes]} \\
&= \sin^2 x + \sin x\cos x + \cos^2 x \quad \text{[Simplifying]} \\
&= (\sin^2 x + \cos^2 x) + \sin x\cos x \quad \text{[Group terms]} \\
&= 1 + \sin x\cos x \quad \text{[Pythagorean Id]} \\
&= \text{RHS}
\end{aligned}$$

■

85. $\frac{\cos A}{1-\tan A} + \frac{\sin A}{1-\cot A} = \sin A + \cos A$

$$\begin{aligned}
\text{LHS} &= \frac{\cos A}{1 - \frac{\sin A}{\cos A}} + \frac{\sin A}{1 - \frac{\cos A}{\sin A}} \quad \text{[Definitions]} \\
&= \frac{\cos A}{\frac{\cos A - \sin A}{\cos A}} + \frac{\sin A}{\frac{\sin A - \cos A}{\sin A}} \quad \text{[Common denominators]} \\
&= \frac{\cos^2 A}{\cos A - \sin A} + \frac{\sin^2 A}{\sin A - \cos A} \quad \text{[Simplify fractions]} \\
&= \frac{\cos^2 A}{\cos A - \sin A} - \frac{\sin^2 A}{\cos A - \sin A} \quad \text{[Adjust sign]} \\
&= \frac{\cos^2 A - \sin^2 A}{\cos A - \sin A} \quad \text{[Combine fractions]} \\
&= \frac{(\cos A - \sin A)(\cos A + \sin A)}{\cos A - \sin A} \quad \text{[Difference of squares]} \\
&= \cos A + \sin A \\
&= \sin A + \cos A \\
&= \text{RHS}
\end{aligned}$$

■

86. $\frac{\sec A-\tan A}{\sec A+\tan A}=\left(\frac{\cos A}{1+\sin A}\right)^2$

$$\begin{aligned}
\text{LHS} &= \frac{\sec A-\tan A}{\sec A+\tan A}\\
&= \frac{\frac{1}{\cos A}-\frac{\sin A}{\cos A}}{\frac{1}{\cos A}+\frac{\sin A}{\cos A}} \quad \text{[Definitions]}\\
&= \frac{\frac{1-\sin A}{\cos A}}{\frac{1+\sin A}{\cos A}}\\
&= \frac{1-\sin A}{1+\sin A} \quad \text{[Simplify fraction]}\\
\text{RHS} &= \left(\frac{\cos A}{1+\sin A}\right)^2\\
&= \frac{\cos^2 A}{(1+\sin A)^2}\\
&= \frac{1-\sin^2 A}{(1+\sin A)^2} \quad \text{[Pythagorean Id]}\\
&= \frac{(1-\sin A)(1+\sin A)}{(1+\sin A)^2} \quad \text{[Difference of squares]}\\
&= \frac{1-\sin A}{1+\sin A} \quad \text{[Simplifying]}\\
\text{LHS} &= \text{RHS}
\end{aligned}$$

■

87. $\frac{(1+\cot^2\theta)\tan\theta}{\sec^2\theta}=\cot\theta$

$$\begin{aligned}
\text{LHS} &= \frac{(1+\cot^2\theta)\tan\theta}{\sec^2\theta}\\
&= \frac{(\operatorname{cosec}^2\theta)\tan\theta}{\sec^2\theta} \quad \text{[Pythagorean Id]}\\
&= \frac{(\frac{1}{\sin^2\theta})(\frac{\sin\theta}{\cos\theta})}{\frac{1}{\cos^2\theta}} \quad \text{[Definitions]}\\
&= \frac{\frac{1}{\sin\theta\cos\theta}}{\frac{1}{\cos^2\theta}}\\
&= \frac{1}{\sin\theta\cos\theta}\cdot\frac{\cos^2\theta}{1} \quad \text{[Fraction division]}\\
&= \frac{\cos\theta}{\sin\theta} \quad \text{[Simplifying]}\\
&= \cot\theta \quad \text{[Definition]}\\
&= \text{RHS}
\end{aligned}$$

■

88. $\frac{\sec\theta+\operatorname{cosec}\theta}{\tan\theta+\cot\theta}=\sin\theta+\cos\theta$

$$\begin{aligned}
\text{LHS} &= \frac{\sec\theta+\operatorname{cosec}\theta}{\tan\theta+\cot\theta}\\
&= \frac{\frac{1}{\cos\theta}+\frac{1}{\sin\theta}}{\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}} \quad \text{[Definitions]}\\
&= \frac{\frac{\sin\theta+\cos\theta}{\cos\theta\sin\theta}}{\frac{\sin^2\theta+\cos^2\theta}{\cos\theta\sin\theta}} \quad \text{[Common denominators]}\\
&= \frac{\frac{\sin\theta+\cos\theta}{\cos\theta\sin\theta}}{\frac{1}{\cos\theta\sin\theta}} \quad \text{[Pythagorean Id]}\\
&= \sin\theta+\cos\theta \quad \text{[Simplify fraction]}\\
&= \text{RHS}
\end{aligned}$$

■

89. $\frac{\tan\theta}{1-\cot\theta} + \frac{\cot\theta}{1-\tan\theta} = 1 + \sec\theta\,\text{cosec}\,\theta$

$$\begin{aligned}
\text{LHS} &= \frac{\tan\theta}{1-\frac{1}{\tan\theta}} + \frac{\frac{1}{\tan\theta}}{1-\tan\theta}\\
&= \frac{\tan\theta}{\frac{\tan\theta-1}{\tan\theta}} + \frac{1}{\tan\theta(1-\tan\theta)}\\
&= \frac{\tan^2\theta}{\tan\theta-1} - \frac{1}{\tan\theta(\tan\theta-1)}\\
&= \frac{\tan^3\theta-1}{\tan\theta(\tan\theta-1)}\\
&= \frac{(\tan\theta-1)(\tan^2\theta+\tan\theta+1)}{\tan\theta(\tan\theta-1)} \quad \text{[Diff cubes]}\\
&= \frac{\tan^2\theta+\tan\theta+1}{\tan\theta}\\
&= \tan\theta + 1 + \cot\theta\\
&= \frac{\sin\theta}{\cos\theta} + 1 + \frac{\cos\theta}{\sin\theta}\\
&= \frac{\sin^2\theta+\sin\theta\cos\theta+\cos^2\theta}{\sin\theta\cos\theta}\\
&= \frac{(\sin^2\theta+\cos^2\theta)+\sin\theta\cos\theta}{\sin\theta\cos\theta}\\
&= \frac{1+\sin\theta\cos\theta}{\sin\theta\cos\theta}\\
&= \frac{1}{\sin\theta\cos\theta} + \frac{\sin\theta\cos\theta}{\sin\theta\cos\theta}\\
&= \text{cosec}\,\theta\sec\theta + 1\\
&= 1 + \sec\theta\,\text{cosec}\,\theta = \text{RHS}
\end{aligned}$$

■

90. $(\text{cosec}\,A - \sin A)(\sec A - \cos A)(\tan A + \cot A) = 1$

$$\begin{aligned}
\text{LHS} &= \left(\frac{1}{\sin A} - \sin A\right)\left(\frac{1}{\cos A} - \cos A\right)\left(\frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}\right)\\
&= \left(\frac{1-\sin^2 A}{\sin A}\right)\left(\frac{1-\cos^2 A}{\cos A}\right)\left(\frac{\sin^2 A+\cos^2 A}{\cos A\sin A}\right)\\
&= \left(\frac{\cos^2 A}{\sin A}\right)\left(\frac{\sin^2 A}{\cos A}\right)\left(\frac{1}{\cos A\sin A}\right) \quad \text{[Pythagorean Ids]}\\
&= \frac{\cos^2 A\sin^2 A}{\sin A\cos A}\cdot\frac{1}{\cos A\sin A}\\
&= (\cos A\sin A)\cdot\frac{1}{\cos A\sin A}\\
&= 1 = \text{RHS}
\end{aligned}$$

■

91. $\frac{1+\tan^2\theta}{1+\cot^2\theta} = \left(\frac{1-\tan\theta}{1-\cot\theta}\right)^2 = \tan^2\theta$ Part 1:

$$\begin{aligned}
\frac{1+\tan^2\theta}{1+\cot^2\theta} &= \frac{\sec^2\theta}{\text{cosec}^2\theta} \quad \text{[Pythagorean Ids]}\\
&= \frac{1/\cos^2\theta}{1/\sin^2\theta} = \frac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta
\end{aligned}$$

Part 2:

$$\begin{aligned}\left(\frac{1-\tan\theta}{1-\cot\theta}\right)^2 &= \left(\frac{1-\frac{\sin\theta}{\cos\theta}}{1-\frac{\cos\theta}{\sin\theta}}\right)^2 \\ &= \left(\frac{\frac{\cos\theta-\sin\theta}{\cos\theta}}{\frac{\sin\theta-\cos\theta}{\sin\theta}}\right)^2 \\ &= \left(\frac{\cos\theta-\sin\theta}{\cos\theta}\cdot\frac{\sin\theta}{\sin\theta-\cos\theta}\right)^2 \\ &= \left(\frac{-(\sin\theta-\cos\theta)}{\cos\theta}\cdot\frac{\sin\theta}{\sin\theta-\cos\theta}\right)^2 \\ &= \left(-\frac{\sin\theta}{\cos\theta}\right)^2 = (-\tan\theta)^2 = \tan^2\theta\end{aligned}$$

Since both parts equal $\tan^2\theta$, the identity holds.

92. $(1+\cot A-\operatorname{cosec} A)(1+\tan A+\sec A)=2$

$$\begin{aligned}\text{LHS} &= \left(1+\frac{\cos A}{\sin A}-\frac{1}{\sin A}\right)\left(1+\frac{\sin A}{\cos A}+\frac{1}{\cos A}\right) \\ &= \left(\frac{\sin A+\cos A-1}{\sin A}\right)\left(\frac{\cos A+\sin A+1}{\cos A}\right) \\ &= \frac{((\sin A+\cos A)-1)((\sin A+\cos A)+1)}{\sin A\cos A} \\ &= \frac{(\sin A+\cos A)^2-1^2}{\sin A\cos A} \quad \text{[Diff squares]} \\ &= \frac{(\sin^2 A+2\sin A\cos A+\cos^2 A)-1}{\sin A\cos A} \\ &= \frac{(\sin^2 A+\cos^2 A)+2\sin A\cos A-1}{\sin A\cos A} \\ &= \frac{1+2\sin A\cos A-1}{\sin A\cos A} \quad \text{[Pythagorean Id]} \\ &= \frac{2\sin A\cos A}{\sin A\cos A} = 2 = \text{RHS}\end{aligned}$$

■

93. $(\sin x+\operatorname{cosec} x)^2+(\cos x+\sec x)^2=\tan^2 x+\cot^2 x+7$

$$\begin{aligned}\text{LHS} &= (\sin^2 x+2\sin x\operatorname{cosec} x+\operatorname{cosec}^2 x) \\ &\quad +(\cos^2 x+2\cos x\sec x+\sec^2 x) \\ &= (\sin^2 x+2(1)+\operatorname{cosec}^2 x)+(\cos^2 x+2(1)+\sec^2 x) \quad \text{[Reciprocal Ids]} \\ &= (\sin^2 x+\cos^2 x)+2+\operatorname{cosec}^2 x+2+\sec^2 x \\ &= 1+4+(1+\cot^2 x)+(1+\tan^2 x) \quad \text{[Pythagorean Ids]} \\ &= 1+4+1+\cot^2 x+1+\tan^2 x \\ &= 7+\cot^2 x+\tan^2 x \\ &= \tan^2 x+\cot^2 x+7 = \text{RHS}\end{aligned}$$

■

94. $\frac{\sin\theta-2\sin^3\theta}{2\cos^3\theta-\cos\theta}=\tan\theta$

$$\begin{aligned}\text{LHS} &= \frac{\sin\theta(1-2\sin^2\theta)}{\cos\theta(2\cos^2\theta-1)} \quad \text{[Factor]} \\ &= \frac{\sin\theta(\cos(2\theta))}{\cos\theta(\cos(2\theta))} \quad \text{[Double Angle Ids]} \\ &= \frac{\sin\theta}{\cos\theta} \quad \text{[Simplify, assuming } \cos(2\theta)\neq 0\text{]} \\ &= \tan\theta = \text{RHS}\end{aligned}$$

■

95. $\sin^6\theta + \cos^6\theta = 1 - 3\sin^2\theta\cos^2\theta$

$$\begin{aligned}
\text{LHS} &= \sin^6\theta + \cos^6\theta \\
&= (\sin^2\theta)^3 + (\cos^2\theta)^3 \\
&= (\sin^2\theta + \cos^2\theta)((\sin^2\theta)^2 - \sin^2\theta\cos^2\theta + (\cos^2\theta)^2) \quad \text{[Sum of cubes]} \\
&= (1)(\sin^4\theta - \sin^2\theta\cos^2\theta + \cos^4\theta) \\
&= (\sin^4\theta + \cos^4\theta) - \sin^2\theta\cos^2\theta \\
&= ((\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta) - \sin^2\theta\cos^2\theta \\
&= ((1)^2 - 2\sin^2\theta\cos^2\theta) - \sin^2\theta\cos^2\theta \\
&= 1 - 2\sin^2\theta\cos^2\theta - \sin^2\theta\cos^2\theta \\
&= 1 - 3\sin^2\theta\cos^2\theta = \text{RHS}
\end{aligned}$$

■

96. $\sec^6\theta = \tan^6\theta + 3\tan^2\theta\sec^2\theta + 1$

$$\begin{aligned}
\text{LHS} &= \sec^6\theta = (\sec^2\theta)^3 \\
&= (1 + \tan^2\theta)^3 \\
&= 1^3 + 3(1)^2(\tan^2\theta) + 3(1)(\tan^2\theta)^2 + (\tan^2\theta)^3 \quad \text{[Binomial expansion]} \\
&= 1 + 3\tan^2\theta + 3\tan^4\theta + \tan^6\theta \\
\text{RHS} &= \tan^6\theta + 3\tan^2\theta\sec^2\theta + 1 \\
&= \tan^6\theta + 3\tan^2\theta(1 + \tan^2\theta) + 1 \\
&= \tan^6\theta + 3\tan^2\theta + 3\tan^4\theta + 1 \\
\text{LHS} &= \text{RHS}
\end{aligned}$$

■

97. $\operatorname{cosec}^6\theta = \cot^6\theta + 3\cot^2\theta\operatorname{cosec}^2\theta + 1$

$$\begin{aligned}
\text{LHS} &= \operatorname{cosec}^6\theta = (\operatorname{cosec}^2\theta)^3 \\
&= (1 + \cot^2\theta)^3 \\
&= 1^3 + 3(1)^2(\cot^2\theta) + 3(1)(\cot^2\theta)^2 + (\cot^2\theta)^3 \quad \text{[Binomial expansion]} \\
&= 1 + 3\cot^2\theta + 3\cot^4\theta + \cot^6\theta \\
\text{RHS} &= \cot^6\theta + 3\cot^2\theta\operatorname{cosec}^2\theta + 1 \\
&= \cot^6\theta + 3\cot^2\theta(1 + \cot^2\theta) + 1 \\
&= \cot^6\theta + 3\cot^2\theta + 3\cot^4\theta + 1 \\
\text{LHS} &= \text{RHS}
\end{aligned}$$

■

98. $(1 + \tan^2 A) + (1 + \frac{1}{\tan^2 A}) = \frac{1}{\sin^2 A - \sin^4 A}$

$$\begin{aligned}
\text{LHS} &= (1 + \tan^2 A) + (1 + \cot^2 A) \\
&= \sec^2 A + \operatorname{cosec}^2 A \quad \text{[Pythagorean Ids]} \\
&= \frac{1}{\cos^2 A} + \frac{1}{\sin^2 A} \\
&= \frac{\sin^2 A + \cos^2 A}{\cos^2 A\sin^2 A} = \frac{1}{\cos^2 A\sin^2 A} \\
\text{RHS} &= \frac{1}{\sin^2 A - \sin^4 A} \\
&= \frac{1}{\sin^2 A(1 - \sin^2 A)} \quad \text{[Factor]} \\
&= \frac{1}{\sin^2 A\cos^2 A} \quad \text{[Pythagorean Id]} \\
\text{LHS} &= \text{RHS}
\end{aligned}$$

■

99. $\frac{\tan A+\tan B}{\cot A+\cot B} = \tan A \tan B$

$$\begin{aligned}\text{LHS} &= \frac{\tan A + \tan B}{\frac{1}{\tan A} + \frac{1}{\tan B}} \\ &= \frac{\tan A + \tan B}{\frac{\tan B+\tan A}{\tan A \tan B}} \\ &= (\tan A + \tan B) \cdot \frac{\tan A \tan B}{\tan A + \tan B} \\ &= \tan A \tan B = \text{RHS}\end{aligned}$$ ■

100. $\cot^2 A \operatorname{cosec}^2 B - \cot^2 B \operatorname{cosec}^2 A = \cot^2 A - \cot^2 B$

$$\begin{aligned}\text{LHS} &= \cot^2 A(1 + \cot^2 B) - \cot^2 B(1 + \cot^2 A) \quad \text{[Pythagorean Ids]} \\ &= (\cot^2 A + \cot^2 A \cot^2 B) - (\cot^2 B + \cot^2 B \cot^2 A) \\ &= \cot^2 A + \cot^2 A \cot^2 B - \cot^2 B - \cot^2 B \cot^2 A \\ &= \cot^2 A - \cot^2 B = \text{RHS}\end{aligned}$$ ■

101. $\tan^2 A \sec^2 B - \tan^2 B \sec^2 A = \tan^2 A - \tan^2 B$

$$\begin{aligned}\text{LHS} &= \tan^2 A(1 + \tan^2 B) - \tan^2 B(1 + \tan^2 A) \quad \text{[Pythagorean Ids]} \\ &= (\tan^2 A + \tan^2 A \tan^2 B) - (\tan^2 B + \tan^2 B \tan^2 A) \\ &= \tan^2 A + \tan^2 A \tan^2 B - \tan^2 B - \tan^2 B \tan^2 A \\ &= \tan^2 A - \tan^2 B = \text{RHS}\end{aligned}$$ ■

102. $\sin^2 x \cos^2 y - \cos^2 x \sin^2 y = \sin^2 x - \sin^2 y$

$$\begin{aligned}\text{LHS} &= \sin^2 x(1 - \sin^2 y) - (1 - \sin^2 x) \sin^2 y \quad \text{[Pythagorean Ids]} \\ &= (\sin^2 x - \sin^2 x \sin^2 y) - (\sin^2 y - \sin^2 x \sin^2 y) \\ &= \sin^2 x - \sin^2 x \sin^2 y - \sin^2 y + \sin^2 x \sin^2 y \\ &= \sin^2 x - \sin^2 y = \text{RHS}\end{aligned}$$ ■

103. $\frac{\cos A \cot A - \sin A \tan A}{\operatorname{cosec} A - \sec A} = 1 + \sin A \cos A$

$$\begin{aligned}\text{Num} &= \cos A(\frac{\cos A}{\sin A}) - \sin A(\frac{\sin A}{\cos A}) \\ &= \frac{\cos^2 A}{\sin A} - \frac{\sin^2 A}{\cos A} = \frac{\cos^3 A - \sin^3 A}{\sin A \cos A} \\ \text{Denom} &= \frac{1}{\sin A} - \frac{1}{\cos A} = \frac{\cos A - \sin A}{\sin A \cos A} \\ \text{LHS} &= \frac{\text{Num}}{\text{Denom}} = \frac{(\cos^3 A - \sin^3 A)/(\sin A \cos A)}{(\cos A - \sin A)/(\sin A \cos A)} \\ &= \frac{\cos^3 A - \sin^3 A}{\cos A - \sin A} \\ &= \frac{(\cos A - \sin A)(\cos^2 A + \cos A \sin A + \sin^2 A)}{\cos A - \sin A} \quad \text{[Diff cubes]} \\ &= \cos^2 A + \cos A \sin A + \sin^2 A \\ &= (\cos^2 A + \sin^2 A) + \sin A \cos A \\ &= 1 + \sin A \cos A = \text{RHS}\end{aligned}$$ ■

104. $(\sec A + \tan A - 1)(\sec A - \tan A + 1) = 2 \tan A$

$$\begin{aligned}\text{LHS} &= (\sec A + (\tan A - 1))(\sec A - (\tan A - 1)) \\ &= \sec^2 A - (\tan A - 1)^2 \quad \text{[Difference of squares]} \\ &= \sec^2 A - (\tan^2 A - 2 \tan A + 1) \\ &= \sec^2 A - \tan^2 A + 2 \tan A - 1 \\ &= (1 + \tan^2 A) - \tan^2 A + 2 \tan A - 1 \quad \text{[Pythagorean Id]} \\ &= 1 + 2 \tan A - 1 \\ &= 2 \tan A = \text{RHS}\end{aligned}$$ ■

105. $(\text{cosec } A + \cot A - 1)(\text{cosec } A - \cot A + 1) = 2\cot A$

$$\begin{aligned}
\text{LHS} &= (\text{cosec } A + (\cot A - 1))(\text{cosec } A - (\cot A - 1)) \\
&= \text{cosec}^2 A - (\cot A - 1)^2 \quad \text{[Difference of squares]} \\
&= \text{cosec}^2 A - (\cot^2 A - 2\cot A + 1) \\
&= \text{cosec}^2 A - \cot^2 A + 2\cot A - 1 \\
&= (1 + \cot^2 A) - \cot^2 A + 2\cot A - 1 \quad \text{[Pythagorean Id]} \\
&= 1 + 2\cot A - 1 \\
&= 2\cot A = \text{RHS}
\end{aligned}$$ ■

106. $2(\sin^6 x + \cos^6 x) - 3(\sin^4 x + \cos^4 x) + 1 = 0$

$$\begin{aligned}
\text{LHS} &= 2(1 - 3\sin^2 x\cos^2 x) - 3(1 - 2\sin^2 x\cos^2 x) + 1 \quad \text{[Using results 97, 75]} \\
&= (2 - 6\sin^2 x\cos^2 x) - (3 - 6\sin^2 x\cos^2 x) + 1 \\
&= 2 - 6\sin^2 x\cos^2 x - 3 + 6\sin^2 x\cos^2 x + 1 \\
&= (2 - 3 + 1) + (-6\sin^2 x\cos^2 x + 6\sin^2 x\cos^2 x) \\
&= 0 + 0 = 0 = \text{RHS}
\end{aligned}$$ ■

107. $\sin^8 x - \cos^8 x = (\sin^2 x - \cos^2 x)(1 - 2\sin^2 x\cos^2 x)$

$$\begin{aligned}
\text{LHS} &= \sin^8 x - \cos^8 x \\
&= (\sin^4 x)^2 - (\cos^4 x)^2 \\
&= (\sin^4 x - \cos^4 x)(\sin^4 x + \cos^4 x) \quad \text{[Difference of squares]} \\
&= ((\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x))(1 - 2\sin^2 x\cos^2 x) \quad \text{[Diff squares, result 75]} \\
&= ((\sin^2 x - \cos^2 x)(1))(1 - 2\sin^2 x\cos^2 x) \quad \text{[Pythagorean Id]} \\
&= (\sin^2 x - \cos^2 x)(1 - 2\sin^2 x\cos^2 x) = \text{RHS}
\end{aligned}$$ ■

108. $\frac{\cos A\,\text{cosec } A - \sin A\sec A}{\cos A + \sin A} = \text{cosec } A - \sec A$

$$\begin{aligned}
\text{Num} &= \cos A(\frac{1}{\sin A}) - \sin A(\frac{1}{\cos A}) \\
&= \frac{\cos^2 A - \sin^2 A}{\sin A\cos A} \\
\text{LHS} &= \frac{\text{Num}}{\text{Denom}} = \frac{(\cos^2 A - \sin^2 A)/(\sin A\cos A)}{\cos A + \sin A} \\
&= \frac{(\cos A - \sin A)(\cos A + \sin A)}{\sin A\cos A(\cos A + \sin A)} \\
&= \frac{\cos A - \sin A}{\sin A\cos A} \\
\text{RHS} &= \text{cosec } A - \sec A = \frac{1}{\sin A} - \frac{1}{\cos A} \\
&= \frac{\cos A - \sin A}{\sin A\cos A} \\
\text{LHS} &= \text{RHS}
\end{aligned}$$ ■

109. $\frac{\tan A + \sec A - 1}{\tan A - \sec A + 1} = \frac{1 + \sin A}{\cos A}$

$$\begin{aligned}
\text{LHS} &= \frac{\tan A + \sec A - (\sec^2 A - \tan^2 A)}{\tan A - \sec A + 1} \quad \text{[Using } 1 = \sec^2 A - \tan^2 A\text{]} \\
&= \frac{\tan A + \sec A - (\sec A - \tan A)(\sec A + \tan A)}{\tan A - \sec A + 1} \\
&= \frac{(\sec A + \tan A)[1 - (\sec A - \tan A)]}{\tan A - \sec A + 1} \quad \text{[Factor]} \\
&= \frac{(\sec A + \tan A)(1 - \sec A + \tan A)}{\tan A - \sec A + 1} \\
&= \sec A + \tan A \quad \text{[Simplifying]} \\
&= \frac{1}{\cos A} + \frac{\sin A}{\cos A} = \frac{1 + \sin A}{\cos A} = \text{RHS}
\end{aligned}$$ ■

110. $\frac{\tan^2\theta}{\sec^2\theta} + \frac{\cot^2\theta}{\text{cosec}^2\theta} = 1$

$$\begin{aligned} \text{LHS} &= \frac{\sin^2\theta/\cos^2\theta}{1/\cos^2\theta} + \frac{\cos^2\theta/\sin^2\theta}{1/\sin^2\theta} \quad \text{[Definitions]} \\ &= \sin^2\theta + \cos^2\theta \quad \text{[Simplify fractions]} \\ &= 1 \quad \text{[Pythagorean Id]} \\ &= \text{RHS} \end{aligned}$$ ■

111. If $\tan\theta + \sin\theta = m$ and $\tan\theta - \sin\theta = n$, prove that $m^2 - n^2 = 4\sqrt{mn}$

$$\begin{aligned} m^2 - n^2 &= (m-n)(m+n) \\ &= ((\tan\theta + \sin\theta) - (\tan\theta - \sin\theta))((\tan\theta + \sin\theta) + (\tan\theta - \sin\theta)) \\ &= (2\sin\theta)(2\tan\theta) = 4\sin\theta\tan\theta \\ mn &= (\tan\theta + \sin\theta)(\tan\theta - \sin\theta) \\ &= \tan^2\theta - \sin^2\theta \\ &= \frac{\sin^2\theta}{\cos^2\theta} - \sin^2\theta = \sin^2\theta\left(\frac{1}{\cos^2\theta} - 1\right) \\ &= \sin^2\theta(\sec^2\theta - 1) = \sin^2\theta\tan^2\theta \\ \sqrt{mn} &= \sqrt{\sin^2\theta\tan^2\theta} = |\sin\theta\tan\theta| \\ 4\sqrt{mn} &= 4|\sin\theta\tan\theta| \end{aligned}$$

Assuming $\sin\theta\tan\theta \geq 0$, then $m^2 - n^2 = 4\sqrt{mn}$.

112. If $x = a\sec\theta + b\tan\theta$ and $y = a\tan\theta + b\sec\theta$, prove that $x^2 - y^2 = a^2 - b^2$

$$\begin{aligned} x^2 &= (a\sec\theta + b\tan\theta)^2 \\ &= a^2\sec^2\theta + 2ab\sec\theta\tan\theta + b^2\tan^2\theta \\ y^2 &= (a\tan\theta + b\sec\theta)^2 \\ &= a^2\tan^2\theta + 2ab\tan\theta\sec\theta + b^2\sec^2\theta \\ x^2 - y^2 &= (a^2\sec^2\theta + 2ab\sec\theta\tan\theta + b^2\tan^2\theta) \\ &\quad - (a^2\tan^2\theta + 2ab\tan\theta\sec\theta + b^2\sec^2\theta) \\ &= a^2\sec^2\theta - b^2\sec^2\theta + b^2\tan^2\theta - a^2\tan^2\theta \\ &= (a^2 - b^2)\sec^2\theta - (a^2 - b^2)\tan^2\theta \\ &= (a^2 - b^2)(\sec^2\theta - \tan^2\theta) \\ &= (a^2 - b^2)(1) = a^2 - b^2 \quad \text{[Pythagorean Id]} \end{aligned}$$ ■

113. If $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$, prove that $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$ Given: $\sin\theta = \sqrt{2}\cos\theta - \cos\theta = (\sqrt{2} - 1)\cos\theta$ Multiply by $(\sqrt{2} + 1)$:

$$\begin{aligned} (\sqrt{2} + 1)\sin\theta &= (\sqrt{2} + 1)(\sqrt{2} - 1)\cos\theta \\ \sqrt{2}\sin\theta + \sin\theta &= (2 - 1)\cos\theta \quad \text{[Difference of squares]} \\ \sqrt{2}\sin\theta + \sin\theta &= \cos\theta \\ \cos\theta - \sin\theta &= \sqrt{2}\sin\theta \quad \text{[Rearranging]} \end{aligned}$$ ■

114. Prove that $(1+\cot A+\tan A)(\sin A-\cos A)=\frac{\sec A}{\operatorname{cosec}^2 A}-\frac{\operatorname{cosec} A}{\sec^2 A}$

$$\begin{aligned}
\text{LHS} &= \left(1+\frac{\cos A}{\sin A}+\frac{\sin A}{\cos A}\right)(\sin A-\cos A)\\
&= \left(\frac{\sin A\cos A+\cos^2 A+\sin^2 A}{\sin A\cos A}\right)(\sin A-\cos A)\\
&= \left(\frac{\sin A\cos A+1}{\sin A\cos A}\right)(\sin A-\cos A)\\
\text{RHS} &= \frac{\sec A}{\operatorname{cosec}^2 A}-\frac{\operatorname{cosec} A}{\sec^2 A}\\
&= \frac{1/\cos A}{1/\sin^2 A}-\frac{1/\sin A}{1/\cos^2 A}\\
&= \frac{\sin^2 A}{\cos A}-\frac{\cos^2 A}{\sin A}\\
&= \frac{\sin^3 A-\cos^3 A}{\sin A\cos A}\\
&= \frac{(\sin A-\cos A)(\sin^2 A+\sin A\cos A+\cos^2 A)}{\sin A\cos A} \quad \text{[Diff cubes]}\\
&= \frac{(\sin A-\cos A)(1+\sin A\cos A)}{\sin A\cos A}\\
\text{LHS} &= \text{RHS} \quad \blacksquare
\end{aligned}$$

(Note: Comparing the factorized forms shows equality)

115. If $\operatorname{cosec}\theta-\sin\theta=a^3$ and $\sec\theta-\cos\theta=b^3$, prove that $a^2b^2(a^2+b^2)=1$

$$\begin{aligned}
a^3 &= \frac{1}{\sin\theta}-\sin\theta=\frac{1-\sin^2\theta}{\sin\theta}=\frac{\cos^2\theta}{\sin\theta}\\
b^3 &= \frac{1}{\cos\theta}-\cos\theta=\frac{1-\cos^2\theta}{\cos\theta}=\frac{\sin^2\theta}{\cos\theta}\\
a^2 &= (a^3)^{2/3}=\left(\frac{\cos^2\theta}{\sin\theta}\right)^{2/3}=\frac{\cos^{4/3}\theta}{\sin^{2/3}\theta}\\
b^2 &= (b^3)^{2/3}=\left(\frac{\sin^2\theta}{\cos\theta}\right)^{2/3}=\frac{\sin^{4/3}\theta}{\cos^{2/3}\theta}\\
a^2b^2 &= \frac{\cos^{4/3}\theta}{\sin^{2/3}\theta}\cdot\frac{\sin^{4/3}\theta}{\cos^{2/3}\theta}=\cos^{2/3}\theta\sin^{2/3}\theta\\
a^2+b^2 &= \frac{\cos^{4/3}\theta}{\sin^{2/3}\theta}+\frac{\sin^{4/3}\theta}{\cos^{2/3}\theta}\\
&= \frac{\cos^2\theta+\sin^2\theta}{\sin^{2/3}\theta\cos^{2/3}\theta}=\frac{1}{(\sin\theta\cos\theta)^{2/3}}\\
a^2b^2(a^2+b^2) &= (\cos\theta\sin\theta)^{2/3}\cdot\frac{1}{(\sin\theta\cos\theta)^{2/3}}\\
&= 1 \quad \text{[Simplifying]} \quad \blacksquare
\end{aligned}$$

116. Prove that $\frac{(1-\sin\theta+\cos\theta)^2}{(1+\cos\theta)(1-\sin\theta)}=2$

$$\begin{aligned}
\text{Num} &= ((1+\cos\theta)-\sin\theta)^2\\
&= (1+\cos\theta)^2-2(1+\cos\theta)\sin\theta+\sin^2\theta\\
&= (1+2\cos\theta+\cos^2\theta)-2\sin\theta-2\sin\theta\cos\theta+\sin^2\theta\\
&= 1+2\cos\theta+(\cos^2\theta+\sin^2\theta)-2\sin\theta-2\sin\theta\cos\theta\\
&= 1+2\cos\theta+1-2\sin\theta-2\sin\theta\cos\theta\\
&= 2+2\cos\theta-2\sin\theta-2\sin\theta\cos\theta\\
&= 2(1+\cos\theta)-2\sin\theta(1+\cos\theta)\\
&= 2(1+\cos\theta)(1-\sin\theta)\\
\text{LHS} &= \frac{2(1+\cos\theta)(1-\sin\theta)}{(1+\cos\theta)(1-\sin\theta)}\\
&= 2 \quad \text{[Simplifying]} \quad \blacksquare
\end{aligned}$$