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Please attempt all the problems on your own first, giving them your best and most honest effort. These solutions are to help you get unstuck on a problem after you have already tried your best.

Your effort and dedication are the true keys to success.

Solutions to JEE CT-02

Date of Exam: 29 June

Syllabus: Sets & Trigonometric Ratios & Identities (till Two IMP Identities)

Topic: Sets & Trigonometry

Sub: Mathematics **CT-02** Prof. Chetan Sir

SECTION-A

51. Let A and B be two sets in the same universal set. Then, $A - B =$

(A) $A \cap B$ (B) $A \cup B$ (C) $A \cap B'$ (D) $A' \cap B$

Solution:

The operation $A - B$ represents the set of elements that are in A but not in B. An element x belongs to $A - B$ if and only if $x \in A$ and $x \notin B$. The condition $x \notin B$ is equivalent to $x \in B'$, where B' is the complement of B. Therefore, an element is in $A - B$ if it is in A and in B'. This is the definition of the intersection of A and B'.

$$A - B = \mathbf{A \cap B'}$$

The correct option is **(C)**.

52. If $\tan A + \cot A = 4$, then $\tan^4 A + \cot^4 A$ is equal to

(A) 110 (B) 191 (C) 80 (D) 194

Solution:

Given $\tan A + \cot A = 4$. Squaring both sides:

$$\begin{aligned}
(\tan A + \cot A)^2 &= 4^2 \\
\tan^2 A + 2\tan A \cot A + \cot^2 A &= 16 \\
\tan^2 A + 2(1) + \cot^2 A &= 16 \quad [\text{Since } \tan A \cot A = 1] \\
\tan^2 A + \cot^2 A &= 14.
\end{aligned}$$

Now, squaring the new equation:

$$\begin{aligned}
(\tan^2 A + \cot^2 A)^2 &= 14^2 \\
\tan^4 A + 2\tan^2 A \cot^2 A + \cot^4 A &= 196 \\
\tan^4 A + 2(1) + \cot^4 A &= 196 \qquad [\text{Since } \tan^2 A \cot^2 A = 1] \\
\tan^4 A + \cot^4 A &= 196 - 2 = \mathbf{194}.
\end{aligned}$$

The correct option is **(D)**.

53. The number of subsets of a set containing n elements is:

(A) n (B) $2^n - 1$ (C) n^2 (D) 2^n

Solution:
Let A be a set with n elements. A subset is formed by choosing some (or none, or all) of the elements from A. For each of the n elements, we have two choices: either include it in the subset or not. Since there are n elements, and for each element there are 2 independent choices, the total number of possible subsets is the product of these choices.

$$\text{Total subsets} = 2 \times 2 \times \cdots \times 2 \text{ (n times)} = \mathbf{2^n}.$$

The correct option is **(D)**.

54. The value of $\frac{\cot 54^\circ}{\tan 36^\circ} + \frac{\tan 20^\circ}{\cot 70^\circ}$ is

(A) 2 (B) 3 (C) 1 (D) 0

Solution:
We use the complementary angle identities:
$\tan(90^\circ - \theta) = \cot\theta$ and $\cot(90^\circ - \theta) = \tan\theta$.
For the first term we convert the denominator $\tan 36^\circ = \tan(90^\circ - 54^\circ) = \cot 54^\circ$.
For the second term, we convert the denominator: $\cot 70^\circ = \cot(90^\circ - 20^\circ) = \tan 20^\circ$.
Now substitute these back into the expression:

$$\begin{aligned}\frac{\cot 54^\circ}{\cot 54^\circ} + \frac{\tan 20^\circ}{\tan 20^\circ} &= 1 + 1 \\ &= \mathbf{2}\end{aligned}$$

The correct option is **(A)**.

55. The symmetric difference of $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$ is

(A) $\{1, 2\}$ (B) $\{1, 2, 4, 5\}$ (C) $\{4, 5\}$ (D) $\{1, 2, 3, 4, 5\}$

Solution:
The symmetric difference of two sets A and B, denoted $A\Delta B$, is the set of elements which are in either of the sets, but not in their intersection. The formula is $A\Delta B = (A - B) \cup (B - A)$.
First, find $A - B$ (elements in A but not in B): $A - B = \{1, 2, 3\} - \{3, 4, 5\} = \{1, 2\}$.
Next, find $B - A$ (elements in B but not in A): $B - A = \{3, 4, 5\} - \{1, 2, 3\} = \{4, 5\}$.
Finally, find the union of these two sets:

$$A\Delta B = \{1, 2\} \cup \{4, 5\} = \{\mathbf{1, 2, 4, 5}\}.$$

The correct option is **(B)**.

56. If $\tan\theta - \cot\theta = a$ and $\sin\theta + \cos\theta = b$, then $(b^2 - 1)^2(a^2 + 4)$ is equal to

(A) 2 (B) -4 (C) ± 4 (D) 4

Solution:
Given $\sin\theta + \cos\theta = b$. Squaring both sides gives:

$$\begin{aligned}(\sin\theta + \cos\theta)^2 &= b^2 \\ \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta &= b^2 \\ 1 + 2\sin\theta\cos\theta &= b^2 \implies 2\sin\theta\cos\theta = b^2 - 1.\end{aligned}$$

Now consider $a = \tan\theta - \cot\theta = \frac{\sin\theta}{\cos\theta} - \frac{\cos\theta}{\sin\theta} = \frac{\sin^2\theta - \cos^2\theta}{\sin\theta\cos\theta}$.
Let's evaluate $a^2 + 4$:

$$\begin{aligned}
a^2 + 4 &= \left(\frac{\sin^2\theta - \cos^2\theta}{\sin\theta\cos\theta}\right)^2 + 4 \\
&= \frac{(\sin^2\theta - \cos^2\theta)^2 + 4\sin^2\theta\cos^2\theta}{(\sin\theta\cos\theta)^2} \\
&= \frac{\sin^4\theta - 2\sin^2\theta\cos^2\theta + \cos^4\theta + 4\sin^2\theta\cos^2\theta}{(\sin\theta\cos\theta)^2} \\
&= \frac{(\sin^2\theta + \cos^2\theta)^2}{(\sin\theta\cos\theta)^2} = \frac{1}{(\sin\theta\cos\theta)^2}.
\end{aligned}$$

Now, let's find the required expression:

$$\begin{aligned}
(b^2 - 1)^2(a^2 + 4) &= (2\sin\theta\cos\theta)^2 \cdot \frac{1}{(\sin\theta\cos\theta)^2} \\
&= 4\sin^2\theta\cos^2\theta \cdot \frac{1}{\sin^2\theta\cos^2\theta} = \mathbf{4}.
\end{aligned}$$

The correct option is **(D)**.

57. For any two sets A and B, $(A - B) \cup (B - A) =$

(A) $(A - B) \cup A$ (B) $(B - A) \cup B$ (C) $(A \cup B) - (A \cap B)$ (D) $(A \cup B) \cap (A \cap B)$

Solution:
The expression $(A - B) \cup (B - A)$ is the definition of the symmetric difference, $A \Delta B$. This set contains all elements that are in A or in B, but not in both. Let's analyze the options:

(A) $(A - B) \cup A = A$. Not correct.

(B) $(B - A) \cup B = B$. Not correct.

(C) $(A \cup B) - (A \cap B)$ means all elements in the union of A and B, except for the elements that are common to both. This perfectly matches the definition of $A \Delta B$.

(D) $(A \cup B) \cap (A \cap B) = A \cap B$. Not correct.

Therefore, $(\mathbf{A} - \mathbf{B}) \cup (\mathbf{B} - \mathbf{A}) = (\mathbf{A} \cup \mathbf{B}) - (\mathbf{A} \cap \mathbf{B})$.
The correct option is **(C)**.

58. $\cos 5^\circ + \cos 10^\circ + \cos 15^\circ + \cdots + \cos 540^\circ =$

(A) 0 (B) 1 (C) -1 (D) 2

Solution:
We can group the terms in the series and use trigonometric identities.

- Group 1: From $\cos 5^\circ$ to $\cos 175^\circ$. Using $\cos(180^\circ - x) = -\cos x$, pairs like $(\cos 5^\circ + \cos 175^\circ)$ sum to 0. With $\cos 90^\circ = 0$, this group sums to 0.
- Group 2: The term $\cos 180^\circ = -1$.
- Group 3: From $\cos 185^\circ$ to $\cos 355^\circ$. Using $\cos(180^\circ + x) = -\cos x$ and $\cos(360^\circ - x) = \cos x$, pairs like $(\cos 185^\circ + \cos 355^\circ)$ sum to 0. This group sums to 0.
- Group 4: The term $\cos 360^\circ = 1$.

- Group 5: From $\cos 365^\circ$ to $\cos 535^\circ$. Due to periodicity, this is equivalent to Group 1 and sums to 0.
- Group 6: The term $\cos 540^\circ = \cos(360^\circ + 180^\circ) = \cos 180^\circ = -1$.

The total sum is the sum of the values from each group:

$$\text{Total Sum} = 0 + (-1) + 0 + 1 + 0 + (-1) = \mathbf{-1}.$$

The correct option is **(C)**.

59. Let U be the universal set containing 700 elements. If A, B are sub-sets of U such that $n(A) = 200, n(B) = 300$ and $n(A \cap B) = 100$. Then, $n(A' \cap B') =$

(A) 400 (B) 600 (C) 300 (D) None of these

Solution:
We need to find $n(A' \cap B')$. By De Morgan's Law, $A' \cap B' = (A \cup B)'$. So we need to find $n((A \cup B)')$. The number of elements in the complement of a set is the total number of elements in the universal set minus the number of elements in the set itself.

$$n((A \cup B)') = n(U) - n(A \cup B)$$

First, we find $n(A \cup B)$ using the Principle of Inclusion-Exclusion:

$$\begin{aligned} n(A \cup B) &= n(A) + n(B) - n(A \cap B) \\ &= 200 + 300 - 100 = 400. \end{aligned}$$

Now, substitute this value back:

$$n((A \cup B)') = 700 - 400 = \mathbf{300}.$$

The correct option is **(C)**.

60. The value of $\tan 20^\circ + \tan 40^\circ + \sqrt{3} \tan 20^\circ \tan 40^\circ$ is

(A) $\frac{1}{\sqrt{3}}$ (B) $\sqrt{3}$ (C) $-\frac{1}{\sqrt{3}}$ (D) $-\sqrt{3}$

Solution:
We start with the tangent addition formula for $\tan(60^\circ)$:

$$\begin{aligned} \tan(60^\circ) &= \tan(20^\circ + 40^\circ) \\ \sqrt{3} &= \frac{\tan 20^\circ + \tan 40^\circ}{1 - \tan 20^\circ \tan 40^\circ} \end{aligned}$$

Now, cross-multiply and rearrange the equation:

$$\begin{aligned} \sqrt{3}(1 - \tan 20^\circ \tan 40^\circ) &= \tan 20^\circ + \tan 40^\circ \\ \sqrt{3} - \sqrt{3} \tan 20^\circ \tan 40^\circ &= \tan 20^\circ + \tan 40^\circ \\ \sqrt{3} &= \tan 20^\circ + \tan 40^\circ + \sqrt{3} \tan 20^\circ \tan 40^\circ. \end{aligned}$$

The value of the given expression is therefore $\boldsymbol{\sqrt{3}}$.
The correct option is **(B)**.

61. For two sets $A \cup B = A$ iff

(A) $B \subseteq A$ (B) $A \subseteq B$ (C) $A \neq B$ (D) $A = B$

Solution:
The union of two sets, $A \cup B$, contains all elements that are in A, or in B, or in both. We are given that $A \cup B = A$. This equality means that the union of A and B results in the set A itself, introducing no new elements. This can only happen if all elements of set B are already contained within set A. This is the definition of B being a subset of A. Therefore, $A \cup B = A$ if and only if $\mathbf{B} \subseteq \mathbf{A}$.
The correct option is **(A)**.

62. If $\sin(\alpha+\beta) = \frac{3}{5}$ and $\cos(\alpha-\beta) = \frac{12}{13}$, where $\alpha+\beta$ and $\alpha-\beta$ are acute angles, then $\tan 2\beta =$

(A) $\frac{16}{33}$ (B) $\frac{63}{16}$ (C) $\frac{56}{33}$ (D) $\frac{16}{63}$

Solution:
We can express $2\beta = (\alpha+\beta) - (\alpha-\beta)$.
Since $\alpha+\beta$ is acute and $\sin(\alpha+\beta) = \frac{3}{5}$, it's a 3-4-5 right triangle, so $\tan(\alpha+\beta) = \frac{3}{4}$.
Since $\alpha-\beta$ is acute and $\cos(\alpha-\beta) = \frac{12}{13}$, it's a 5-12-13 right triangle, so $\tan(\alpha-\beta) = \frac{5}{12}$.
Now substitute these into the formula for $\tan(2\beta)$:

$$\begin{aligned}
\tan(2\beta) &= \frac{\tan(\alpha+\beta) - \tan(\alpha-\beta)}{1 + \tan(\alpha+\beta)\tan(\alpha-\beta)} \\
&= \frac{\frac{3}{4} - \frac{5}{12}}{1 + (\frac{3}{4})(\frac{5}{12})} \\
&= \frac{\frac{9-5}{12}}{1 + \frac{15}{48}} = \frac{\frac{4}{12}}{\frac{48+15}{48}} \\
&= \frac{1}{3} \times \frac{48}{63} = \mathbf{\frac{16}{63}}.
\end{aligned}$$

The correct option is **(D)**.

63. Suppose $A_1, \ldots, A_{30}$ are thirty sets each with 5 elements and $B_1, \ldots, B_n$ are n sets each with 3 elements. Let $\bigcup_{i=1}^{30} A_i = \bigcup_{j=1}^{n} B_j = S$. If each element of S belongs to exactly 10 of the A_i's and exactly 9 of the B_j's, then n is equal to

(A) 15 (B) 3 (C) 45 (D) 35

Solution:
Let's find the size of the union set S. The sum of the sizes of all A_i sets is $\sum_{i=1}^{30} n(A_i) = 30 \times 5 = 150$. Since each element of S is in exactly 10 of the A_i's, this sum is also equal to $10 \times n(S)$.

$$10 \cdot n(S) = 150 \implies n(S) = 15.$$

Now we do the same for the B_j sets. The sum of their sizes is $\sum_{j=1}^{n} n(B_j) = n \times 3 = 3n$. Since each element of S is in exactly 9 of the B_j's, this sum is equal to $9 \times n(S)$.

$$9 \cdot n(S) = 3n$$

Substitute the value $n(S) = 15$:

$$\begin{aligned}
9 \times 15 &= 3n \\
135 &= 3n \\
n &= \frac{135}{3} = \mathbf{45}.
\end{aligned}$$

The correct option is **(C)**.

64. If $\frac{\pi}{2} < \alpha < \pi, \pi < \beta < \frac{3\pi}{2}; \sin\alpha = \frac{15}{17}$ and $\tan\beta = \frac{12}{5}$ then the value of $\sin(\beta - \alpha)$ is

(A) $-17/221$ (B) $-21/221$ (C) $21/221$ (D) $171/221$

Solution:
We need $\sin(\beta - \alpha) = \sin\beta\cos\alpha - \cos\beta\sin\alpha$.
For α in Quadrant II, sine is positive, cosine is negative.
Given $\sin\alpha = \frac{15}{17}$, we find $\cos\alpha = -\sqrt{1 - (\frac{15}{17})^2} = -\frac{8}{17}$.
For β in Quadrant III, tangent is positive, sine and cosine are negative.
Given $\tan\beta = \frac{12}{5}$, we have a 5-12-13 triangle, so $\sin\beta = -\frac{12}{13}$ and $\cos\beta = -\frac{5}{13}$.
Now substitute these into the formula:

$$\begin{aligned}\sin(\beta - \alpha) &= \left(-\frac{12}{13}\right)\left(-\frac{8}{17}\right) - \left(-\frac{5}{13}\right)\left(\frac{15}{17}\right)\\ &= \frac{96}{221} - \left(-\frac{75}{221}\right)\\ &= \frac{96+75}{221} = \mathbf{\frac{171}{221}}.\end{aligned}$$

The correct option is **(D)**.

65. Two finite sets have m and n elements. The number of subsets of the first set is 240 more than that of the second. The values of m and n are respectively.

(A) 4, 7 (B) 7, 4 (C) 8, 4 (D) 7, 7

Solution:
The number of subsets of a set with k elements is 2^k. Let the sets have m and n elements. We are given:

$$2^m = 2^n + 240 \implies 2^m - 2^n = 240$$

Factor out the smaller power, 2^n: $2^n(2^{m-n} - 1) = 240$. We write 240 as a product of a power of 2 and an odd number: $240 = 16 \times 15 = 2^4 \times 15$. By comparing $2^n(2^{m-n} - 1)$ with $2^4 \times 15$, we can equate the even and odd parts:

- $2^n = 2^4 \implies \mathbf{n = 4}$.
- $2^{m-n} - 1 = 15 \implies 2^{m-n} = 16 = 2^4$. So, $m - n = 4$.

Substituting $n = 4$ into the second result gives $m - 4 = 4$, which means $\mathbf{m = 8}$. The values are m=8 and n=4.
The correct option is **(C)**.

66. The value of the expression $\frac{\sin^2\frac{\pi}{16}+\sin^2\frac{3\pi}{16}+\sin^2\frac{5\pi}{16}+\sin^2\frac{7\pi}{16}}{\cos^2\frac{\pi}{12}+\cos^2\frac{3\pi}{12}+\cos^2\frac{5\pi}{12}+\cos^2\frac{7\pi}{12}+\cos^2\frac{9\pi}{12}+\cos^2\frac{11\pi}{12}}$ is:

(A) 1 (B) $\frac{2}{3}$ (C) $\frac{1}{3}$ (D) 2

Solution:
Let's evaluate the numerator (N) and denominator (D) separately.
For the numerator, we use $\sin(\frac{\pi}{2}-x)=\cos x$.
Note that $\frac{7\pi}{16}=\frac{\pi}{2}-\frac{\pi}{16}$ and $\frac{5\pi}{16}=\frac{\pi}{2}-\frac{3\pi}{16}$.

$$\begin{aligned}N &= \sin^2\frac{\pi}{16}+\sin^2\frac{3\pi}{16}+\sin^2(\frac{\pi}{2}-\frac{3\pi}{16})+\sin^2(\frac{\pi}{2}-\frac{\pi}{16})\\ &= \sin^2\frac{\pi}{16}+\sin^2\frac{3\pi}{16}+\cos^2\frac{3\pi}{16}+\cos^2\frac{\pi}{16}\\ &= (\sin^2\frac{\pi}{16}+\cos^2\frac{\pi}{16})+(\sin^2\frac{3\pi}{16}+\cos^2\frac{3\pi}{16})=1+1=2.\end{aligned}$$

For the denominator, we use $\cos(\pi-x)=-\cos x \implies \cos^2(\pi-x)=\cos^2 x$ and $\cos(\frac{\pi}{2}-x)=\sin x$.

$$\begin{aligned}D &= \cos^2\frac{\pi}{12}+\cos^2\frac{3\pi}{12}+\cos^2\frac{5\pi}{12}+\cos^2(\pi-\frac{5\pi}{12})+\cos^2(\pi-\frac{3\pi}{12})+\cos^2(\pi-\frac{\pi}{12})\\ &= 2\left(\cos^2\frac{\pi}{12}+\cos^2\frac{3\pi}{12}+\cos^2\frac{5\pi}{12}\right)\\ &= 2\left(\cos^2\frac{\pi}{12}+\cos^2\frac{\pi}{4}+\sin^2\frac{\pi}{12}\right) && [\text{Since } \cos\tfrac{5\pi}{12}=\sin\tfrac{\pi}{12}]\\ &= 2\left((\cos^2\frac{\pi}{12}+\sin^2\frac{\pi}{12})+\cos^2\frac{\pi}{4}\right)\\ &= 2\left(1+(\frac{1}{\sqrt{2}})^2\right)\\ &= 2(1+\frac{1}{2})=3.\end{aligned}$$

The value of the expression is $\frac{N}{D}=\mathbf{\frac{2}{3}}$. The correct option is **(B)**.

67. Which of the following are disjoint sets?

(A) $A=\{1,3,5,7,9\}$, $B=\{x : x \text{ is odd number upto } 9\}$

(B) $A=\{2,4,6,8\}$, $B=\{1,3,5,2\}$

(C) $A=\{2,3\}$, $B=\{x : x^2-5x+6=0\}$

(D) $A=\{3,5,9,10\}$, $B=\{1,2,11,12\}$

Solution:
Two sets are disjoint if their intersection is the empty set ($A\cap B=\emptyset$).

(A) $B=\{1,3,5,7,9\}$. Here $A=B$, so $A\cap B=A\neq\emptyset$. Not disjoint.

(B) $A\cap B=\{2\}$. Not disjoint.

(C) Solving $x^2-5x+6=0$ gives $(x-2)(x-3)=0$, so $B=\{2,3\}$. Here $A=B$, so $A\cap B\neq\emptyset$. Not disjoint.

(D) A and B have no common elements. $A\cap B=\emptyset$. These sets are **disjoint**.

The correct option is **(D)**.

68. If $m\tan(\theta-30^\circ)=n\tan(\theta+120^\circ)$ then $\frac{m+n}{m-n}=$

(A) $2\cos 2\theta$ (B) $\cos 2\theta$ (C) $2\sin 2\theta$ (D) $\sin 2\theta$

Solution:
From the given equation, we have $\frac{m}{n}=\frac{\tan(\theta+120^\circ)}{\tan(\theta-30^\circ)}$. Using Componendo and Dividendo:

$$\frac{m+n}{m-n}=\frac{\tan(\theta+120^\circ)+\tan(\theta-30^\circ)}{\tan(\theta+120^\circ)-\tan(\theta-30^\circ)}$$

Convert to sines and cosines:

$$\begin{aligned}\frac{m+n}{m-n}&=\frac{\frac{\sin(\theta+120^\circ)}{\cos(\theta+120^\circ)}+\frac{\sin(\theta-30^\circ)}{\cos(\theta-30^\circ)}}{\frac{\sin(\theta+120^\circ)}{\cos(\theta+120^\circ)}-\frac{\sin(\theta-30^\circ)}{\cos(\theta-30^\circ)}}\\&=\frac{\sin(\theta+120^\circ)\cos(\theta-30^\circ)+\cos(\theta+120^\circ)\sin(\theta-30^\circ)}{\sin(\theta+120^\circ)\cos(\theta-30^\circ)-\cos(\theta+120^\circ)\sin(\theta-30^\circ)}\\&=\frac{\sin((\theta+120^\circ)+(\theta-30^\circ))}{\sin((\theta+120^\circ)-(\theta-30^\circ))} &&\text{[Using }\sin(A\pm B)\text{ formulas]}\\&=\frac{\sin(2\theta+90^\circ)}{\sin(150^\circ)}=\frac{\cos(2\theta)}{1/2}=\mathbf{2}\cos\mathbf{2}\boldsymbol{\theta}.\end{aligned}$$

The correct option is **(A)**.

69. 20 teachers of a school either teach mathematics or physics. 12 of them teach mathematics while 4 teach both the subjects. Then the number of teachers teaching physics only is

(A) 12 (B) 8 (C) 16 (D) 4

Solution:
Let M be the set of teachers who teach Mathematics and P for Physics. Given: $n(M\cup P)=20$, $n(M)=12$, and $n(M\cap P)=4$. We need to find $n(P-M)$. First, find the total number of physics teachers, $n(P)$.

$$\begin{aligned}n(M\cup P)&=n(M)+n(P)-n(M\cap P)\\20&=12+n(P)-4\\n(P)&=20-8=12.\end{aligned}$$

The number of teachers teaching Physics only is:

$$n(P-M)=n(P)-n(P\cap M)=12-4=\mathbf{8}.$$

The correct option is **(B)**.

70. If $\sin^4\theta+\cos^4\theta=a$, then the value of $\sin^6\theta+\cos^6\theta$ in terms of a is:

(A) $\frac{2a-1}{3}$ (B) $a^2-\frac{1}{2}$ (C) $\frac{3a+1}{2}$ (D) $\frac{3a-1}{2}$

Solution:
We are given $\sin^4\theta+\cos^4\theta=a$. We can write this as $(\sin^2\theta+\cos^2\theta)^2-2\sin^2\theta\cos^2\theta=a$, which simplifies

to $1 - 2\sin^2\theta\cos^2\theta = a$. From this, we find $\sin^2\theta\cos^2\theta = \frac{1-a}{2}$. Now we evaluate $\sin^6\theta + \cos^6\theta$:

$$\begin{aligned}
\sin^6\theta + \cos^6\theta &= (\sin^2\theta)^3 + (\cos^2\theta)^3 \\
&= (\sin^2\theta + \cos^2\theta)(\sin^4\theta - \sin^2\theta\cos^2\theta + \cos^4\theta) \\
&= (1)\cdot[(\sin^4\theta + \cos^4\theta) - \sin^2\theta\cos^2\theta] \\
&= a - \sin^2\theta\cos^2\theta \\
&= a - \left(\frac{1-a}{2}\right) \\
&= \frac{2a-(1-a)}{2} \\
&= \frac{\mathbf{3a-1}}{\mathbf{2}}.
\end{aligned}$$

The correct option is **(D)**.

SECTION-B

71. If A, B, C are sets, such that $n(A) = 2, n(B) = 3, n(C) = 4$. If P(X) denotes power set of X and $k = \frac{n(P(P(C)))}{n(P(P(A)))\cdot n(P(B))}$, then k is

Solution:
The number of elements in the power set of a set X with m elements is $n(P(X)) = 2^m$. We have $n(P(A)) = 2^2 = 4$, $n(P(B)) = 2^3 = 8$, and $n(P(C)) = 2^4 = 16$. Next, we find the size of the power set of the power set. $n(P(P(A))) = 2^{n(P(A))} = 2^4 = 16$. $n(P(P(C))) = 2^{n(P(C))} = 2^{16}$. Now we calculate k:

$$\begin{aligned}
k &= \frac{n(P(P(C)))}{n(P(P(A)))\cdot n(P(B))} \\
&= \frac{2^{16}}{16\cdot 8} = \frac{2^{16}}{2^4\cdot 2^3} \\
&= \frac{2^{16}}{2^7} = 2^{16-7} = 2^9 = \mathbf{512}.
\end{aligned}$$

The answer is **512**.

72. In a class of 175 students, the data shows: Mathematics 100; Physics 70; Chemistry 40; Math & Physics 30; Math & Chemistry 28; Physics & Chemistry 23; Math, Physics & Chemistry 18. How many students have offered Mathematics alone?

Solution:
Let M, P, C be the sets of students taking Mathematics, Physics, and Chemistry. We want to find the number of students in M only. This can be found by taking the total number of Math students and subtracting those who also take other subjects.

- Number of students in Math and Physics only: $n(M\cap P) - n(M\cap P\cap C) = 30 - 18 = 12$.
- Number of students in Math and Chemistry only: $n(M\cap C) - n(M\cap P\cap C) = 28 - 18 = 10$.
- Number of students in all three: $n(M\cap P\cap C) = 18$.

The number of students in Mathematics alone is the total in Mathematics minus all the intersection groups involving Mathematics:

$$\begin{aligned} n(M \text{ only}) &= n(M) - [n(M \cap P \text{ only}) + n(M \cap C \text{ only}) + n(M \cap P \cap C)] \\ &= 100 - (12 + 10 + 18) \\ &= 100 - 40 = \mathbf{60}. \end{aligned}$$

The answer is **60**.

73. Let A and B be two sets such that $n(A) = 16, n(B) = 14, n(A \cup B) = 25$. Then, $n(A \cap B)$ is equal to

Solution:
We use the Principle of Inclusion-Exclusion for two sets: $n(A \cup B) = n(A) + n(B) - n(A \cap B)$. Substitute the given values into the formula and solve for $n(A \cap B)$:

$$\begin{aligned} 25 &= 16 + 14 - n(A \cap B) \\ 25 &= 30 - n(A \cap B) \\ n(A \cap B) &= 30 - 25 = \mathbf{5}. \end{aligned}$$

The answer is **5**.

74. The value of $\frac{\sin 24^\circ \cos 6^\circ - \sin 6^\circ \sin 66^\circ}{\sin 21^\circ \cos 39^\circ - \cos 51^\circ \sin 69^\circ} + 8(\sin 163^\circ \cos 347^\circ + \sin 73^\circ \sin 167^\circ)$ is

Solution:
Let's evaluate the fraction and the second term separately.

$$\begin{aligned} \textbf{Numerator} &= \sin 24^\circ \cos 6^\circ - \sin 6^\circ \sin 66^\circ \\ &= \sin 24^\circ \cos 6^\circ - \sin 6^\circ \cos 24^\circ \\ &= \sin(24^\circ - 6^\circ) = \sin(18^\circ). \end{aligned}$$

$$\begin{aligned} \textbf{Denominator} &= \sin 21^\circ \cos 39^\circ - \cos 51^\circ \sin 69^\circ \\ &= \sin 21^\circ \cos 39^\circ - \sin 39^\circ \cos 21^\circ \\ &= \sin(21^\circ - 39^\circ) \\ &= -\sin(18^\circ). \end{aligned}$$

The value of the fraction is $\frac{\sin(18^\circ)}{-\sin(18^\circ)} = -1$.

$$\begin{aligned} \textbf{Second Term} &= 8(\sin 163^\circ \cos 347^\circ + \sin 73^\circ \sin 167^\circ). \\ &\text{[We simplify the angles:]} \\ &= 8(\sin(180 - 17)\cos(360 - 13) + \cos(90 - 17)\sin(180 - 13)) \\ &= 8(\sin 17^\circ \cos 13^\circ + \cos 17^\circ \sin 13^\circ). \\ &\text{[This simplifies using the } \sin(A+B) \text{ formula]} \\ &= 8\sin(17^\circ + 13^\circ) \\ &= 8\sin(30^\circ) = 8(\frac{1}{2}) = 4. \end{aligned}$$

Total value = (Fraction part) + (Second part) = $-1 + 4 = \mathbf{3}$. The answer is **3**.

75. Let α and β be two angles such that $\alpha+\beta=\frac{5\pi}{12}$ and $\alpha-\beta=\frac{\pi}{12}$. Find the value of $4(\cos^2\alpha-\sin^2\beta)$.

Solution:
We use the identity $\cos^2 A-\sin^2 B=\cos(A+B)\cos(A-B)$.

Let $A=\alpha$ and $B=\beta$.

The expression becomes $4\cos(\alpha+\beta)\cos(\alpha-\beta)$.

We are given the values $\alpha+\beta=\frac{5\pi}{12}$ and $\alpha-\beta=\frac{\pi}{12}$.

Substitute these values into the expression:

$$\begin{aligned}\textbf{Required} &= 4\cos\left(\frac{5\pi}{12}\right)\cos\left(\frac{\pi}{12}\right)\\ &= 4\cos\left(75^\circ\right)\cos\left(15^\circ\right)\\ &= 4\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)\\ &= 4(\frac{3-1}{8})\\ &= 1\end{aligned}$$

Method-2

The expression $4\cos(\alpha+\beta)\cos(\alpha-\beta)$. is of the form

$$\begin{aligned}&= 4\cos(\alpha+\beta)\cos(\alpha-\beta)\\ &= 4\cos\left(\frac{5\pi}{12}\right)\cos\left(\frac{\pi}{12}\right)\\ &= 2\cdot[2\cos A\cos B]=2[\cos(A+B)+\cos(A-B)]\\ &= 2\left[2\cos\left(\frac{5\pi}{12}\right)\cos\left(\frac{\pi}{12}\right)\right]\\ &= 2\left[\cos\left(\frac{5\pi}{12}+\frac{\pi}{12}\right)+\cos\left(\frac{5\pi}{12}-\frac{\pi}{12}\right)\right]\\ &= 2\left[\cos\left(\frac{6\pi}{12}\right)+\cos\left(\frac{4\pi}{12}\right)\right]\\ &= 2\left[\cos\left(\frac{\pi}{2}\right)+\cos\left(\frac{\pi}{3}\right)\right]\\ &= 2\left[0+\frac{1}{2}\right]=\mathbf{1}.\end{aligned}$$

The answer is **1**.