Quadratic Equation: Section 1

Basic Questions

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Quadratic Equation

An equation of the form $ax^2 + bx + c = 0$, where a, b, c are real numbers and $a \neq 0$, is called a quadratic equation.

- The values of x that satisfy the equation are called the **roots**.
- The expression $D = b^2 - 4ac$ is known as the **Discriminant**.

Methods to Solve Quadratic Equations

1. **Factorisation**
2. **Sri Dharacharya Method (Quadratic Formula):** The roots of the equation are given by:

$D = b^2 - 4ac$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Relation Between Roots and Coefficients

Sum of the Roots

$ax^2+bx+c=0$ (roots x_1, x_2)

$$x_1 + x_2 = -\frac{b}{a} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$$

Product of the Roots

$\sqrt{x^2} = |x|$

$$x_1 \cdot x_2 = \frac{c}{a} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

Difference of the Roots

$(x_1-x_2)^2 = (x_1+x_2)^2 - 4x_1x_2$

$|x_1-x_2| = \sqrt{\left(-\frac{b}{a}\right)^2 - 4\left(\frac{c}{a}\right)}$

$$|x_1 - x_2| = \frac{\sqrt{b^2-4ac}}{|a|} = \frac{\sqrt{D}}{|a|}$$

Que.1 JEE Main 2024 Ans:(D)

The number of solutions of the equation $e^{\sin x} - 2e^{-\sin x} = 2$ is:

(A) 2 (B) More than 2 (C) 1 (D) 0

$$e^{\sin x} - 2e^{-\sin x} = 2$$

$$e^{\sin x} - 2\frac{1}{e^{\sin x}} = 2$$

let $e^{\sin x} = t$

$$t - \frac{2}{t} = 2$$

$$t^2 - 2 = 2t$$

$$t^2 - 2t - 2 = 0$$

$$t = \frac{2 \pm \sqrt{4 - 4(1)(-2)}}{2(1)} = \frac{2 \pm 2\sqrt{3}}{2}$$

$$t = 1 \pm \sqrt{3}$$

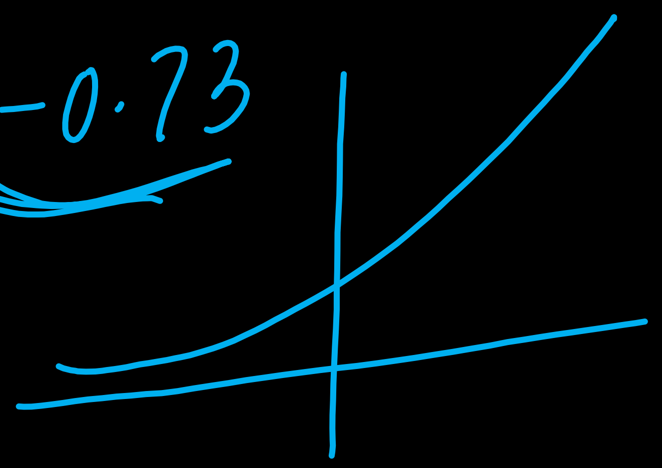
$e^{\sin x} = 1 + \sqrt{3}$ → Not possible

$e^{\sin x} = 2.73$

$e^{[-1,1]}$: $e^{-1} = \frac{1}{e}$, $e^{1} = 2.718$

$e^{\sin x} = 1 - \sqrt{3} = 1 - 1.73 = -0.73$ → Reject

$e^{\sin x} > 0$



Que.2 JEE Main 2023 Ans:(B)

Let $S = \{x \in \mathbb{R} : (\sqrt{3}+\sqrt{2})^{x^2-4} + (\sqrt{3}-\sqrt{2})^{x^2-4} = 10\}$. Then $n(S)$ is equal to:

(A) 2 (B) 4 (C) 6 (D) 0

$$(\sqrt{3}+\sqrt{2})^{x^2-4} + (\sqrt{3}-\sqrt{2})^{x^2-4} = 10$$

$$t \qquad \frac{1}{t}$$

$$(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2}) = 3-2 = 1$$

$$(\sqrt{3}+\sqrt{2}) = \frac{1}{(\sqrt{3}-\sqrt{2})}$$

$$(\sqrt{2}-1)(\sqrt{2}+1) = 1$$

let $(\sqrt{3}+\sqrt{2})^{x^2-4} = t$

$$(\sqrt{3}+\sqrt{2})^{x^2-4} + \left(\frac{1}{\sqrt{3}+\sqrt{2}}\right)^{x^2-4} = 10$$

$$t + \frac{1}{t} = 10$$

$$t^2 - 10t + 1 = 0$$

$$t = \frac{10 \pm \sqrt{100-4}}{2}$$

$$t = \frac{10 \pm 4\sqrt{6}}{2}$$

$$t = 5 \pm 2\sqrt{6}$$

$$\left(\sqrt{3}+\sqrt{2}\right)^{x^2-4} = t$$

$$t = 5 \pm 2\sqrt{6}$$

$$\left(\sqrt{3}+\sqrt{2}\right)^{2} = 5+2\sqrt{6}$$

$$\left(\sqrt{3}+\sqrt{2}\right)^{x^2-4} = 5+2\sqrt{6}$$

$$\left(\sqrt{3}+\sqrt{2}\right)^{x^2-4} = \left(\sqrt{3}+\sqrt{2}\right)^{2}$$

$$x^2-4 = 2$$

$$x^2 = 6$$

$$x = \pm\sqrt{6}$$

$$\left(\sqrt{3}+\sqrt{2}\right)^{x^2-4} = \left(5-2\sqrt{6}\right)$$

$$\left(\sqrt{3}+\sqrt{2}\right)^{x^2-4} = \left(\sqrt{3}-\sqrt{2}\right)^{2}$$

$$\left(\sqrt{3}+\sqrt{2}\right)^{x^2-4} = \left(\sqrt{3}+\sqrt{2}\right)^{-2}$$

$$x^2-4 = -2$$

$$x^2 = 2$$

$$x = \pm\sqrt{2}$$

Que.3 JEE Main 2021 Ans: 1

The number of solutions of the equation

$$32^{\tan^2 x} + 32^{\sec^2 x} = 81, \quad 0 \le x \le \frac{\pi}{4}$$

is:

Ans: 1

$1 + \tan^2 x = \sec^2 x$

$32^{\tan^2 x} + 32^{(1+\tan^2 x)} = 81$

$32^{\tan^2 x} + 32^1 \cdot 32^{\tan^2 x} = 81$

let $32^{\tan^2 x} = t$

$t + 32t = 81$

$33t = 81$

$t = \frac{81}{33} = \frac{27}{11}$

$32^{\tan^2 x} = \frac{27}{11} = 2.45$

$32^{[0,1]} = [1, 32]$

Range

$0 \le x \le \frac{\pi}{4}$

$\Downarrow$

$0 \le \tan x \le 1$

$\Downarrow$

$0 \le \tan^2 x \le 1$

$\tan 45^\circ = 1$, $\tan 0^\circ = 0$, ∞

$\therefore$ It is possible for 1 value of x

Que.4 JEE Main 2025 (24 Jan, Shift-I) Ans:(D)

The product of all the rational roots of the equation

$$(x^2 - 9x + 11)^2 - (x - 4)(x - 5) = 3$$

is equal to:

(A) 28 (B) 21 (C) 7 (D) 14

$(x^2-9x+11)^2 - (x^2-9x+20) = 3$

let $x^2-9x=t$

$(t+11)^2 - (t+20) - 3 = 0$

$t^2+21t+98=0$

$98 = 2 \times 49 = 2 \times 7 \times 7$

$t=-14$ $t=-7$

$x^2-9x=-14$ $x^2-9x=-7$

$x^2-9x+14=0$ $x^2-9x+7=0$

$x=7$ $x=2$ No rational roots

$x_1 x_2 = 7\times 2 = 14$

Que.5 JEE Main 2025 (29 Jan, Shift-I) Ans:(C)

The number of solutions of the equation

$$\left(\frac{9}{x}-\frac{9}{\sqrt{x}}+2\right)\left(\frac{2}{x}-\frac{7}{\sqrt{x}}+3\right)=0$$

is:

(A) 1 (B) 3 (C) 4 (D) 2

let $\frac{1}{\sqrt{x}}=t$

$\frac{1}{x}=t^2$

$(9t^2-9t+2)(2t^2-7t+3)=0$

$9t^2-9t+2=0$ OR $2t^2-7t+3=0$

$D=81-4(9)(2)$

$t=\frac{9\pm3}{18}$ → $2/3$, $1/3$

$2t^2-6t-t+3=0$

$t=\frac{1}{2}$ OR $t=3$

Que.6 JEE Main 2023 (31 Jan, Shift-I) Ans:(B)

The number of real roots of the equation

$$\sqrt{x^2-4x+3}+\sqrt{x^2-9}=\sqrt{4x^2-14x+6}$$

is:

(A) 0 (B) 1 (C) 3 (D) 2

$2(2x^2-7x+3)$

$$\sqrt{(x-3)(x-1)}+\sqrt{(x-3)(x+3)}=\sqrt{2(2x-1)(x-3)}$$

$$\left(\sqrt{x-3}\right)\left(\sqrt{x-1}+\sqrt{x+3}\right)=\left(\sqrt{x-3}\right)\left(\sqrt{2(2x-1)}\right)$$

$$\sqrt{x-1}+\sqrt{x+3}=\sqrt{2(2x-1)}$$

$$2\sqrt{x-1}\sqrt{x+3}=2x-4$$

$(x-1)(x+3)=x^2-4x+4$

$x^2+2x-3=x^2-4x+4$

$6x=7$

$x=7/6$ Reject

$\sqrt{x-3}=0$

$x=3$ → only sol^n

Sum = -10 + 6 + 1 + 4 + 2 = 3

Que.7 JEE Main 2016 Ans:(A)

The sum of all real values of x satisfying the equation

$$(x^2 - 5x + 5)^{x^2+4x-60} = 1$$

is:

(A) 3 (B) -4 (C) 6 (D) 5

$0^0 \rightarrow$ can't be deter

$(\text{Base})^{(\text{Power})} = 1$

Case I : Power = 0 , Base ≠ 0 , R - {0}

Case II : Base = 1 , Power ∈ R

Case III : Base = -1 , Power = even

Case I : $x^2 + 4x - 60 = 0$

$x = -10$, $x = 6$

Case II : $x^2 - 5x + 5 = 1$

$x^2 - 5x + 4 = 0$

$x = 1$, $x = 4$

Case III : $x^2 - 5x + 5 = -1$

$x^2 - 5x + 6 = 0$

$x = 2$, $x = 3$

Power = -48

$(-1)^{-48} = \frac{1}{(-1)^{48}}$

Newton's Method

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Newton's Method: Power of Roots

Newton's Sums Theorem

Let α and β be the roots of the quadratic equation $ax^2 + bx + c = 0$.

and $S_n = \alpha^n + \beta^n$, then

$$aS_n + bS_{n-1} + cS_{n-2} = 0$$

$x^2 \to S_n$

$x \to S_{n-1}$

$1 \to S_{n-2}$

Note: This relation also holds if $S_n = k_1\alpha^n + k_2\beta^n$ *for any constants* k_1, k_2.

e.g. $S_n = 32\alpha^n - 31\beta^n$

Proof

Since α is a root of $ax^2 + bx + c = 0$, we have:

$$a\alpha^2 + b\alpha + c = 0$$

Multiply this entire equation by α^{n-2} (for $n \geq 2$):

$$\alpha^{n-2}(a\alpha^2 + b\alpha + c) = 0$$

$$a\alpha^n + b\alpha^{n-1} + c\alpha^{n-2} = 0 \quad \text{(Equation 1)}$$

Similarly, since β is a root, we have $a\beta^2 + b\beta + c = 0$. Multiply by β^{n-2}:

$$a\beta^n + b\beta^{n-1} + c\beta^{n-2} = 0 \quad \text{(Equation 2)}$$

Now, add Equation 1 and Equation 2:

$$(a\alpha^n + b\alpha^{n-1} + c\alpha^{n-2}) + (a\beta^n + b\beta^{n-1} + c\beta^{n-2}) = 0$$

Group the terms by coefficients a, b, and c:

$$a(\alpha^n + \beta^n) + b(\alpha^{n-1} + \beta^{n-1}) + c(\alpha^{n-2} + \beta^{n-2}) = 0$$

By definition, $S_n = \alpha^n + \beta^n$, so we can substitute to get the final result:

$$aS_n + bS_{n-1} + cS_{n-2} = 0$$

JEE Adv. 2011 / Adv 2015 / Main-2021

Let α and β be the roots of $x^2 - 6x - 2 = 0$ with $\alpha > \beta$. If $a_n = \alpha^n - \beta^n$ for $n \geq 1$, then the value of

$$\frac{a_{10} - 2a_8}{2a_9}$$

is ______.

① $1x^2 - 6x - 2 = 0$ (roots α, β)

② $a_n = \alpha^n - \beta^n \quad (n \geq 1)$

$x^2 \longrightarrow a_n$

$x \longrightarrow a_{n-1}$

$1 \longrightarrow a_{n-2}$

$1.a_n - 6.a_{n-1} - 2\,a_{n-2} = 0$

$n = 10$

$a_{10} - 6a_9 - 2a_8 = 0$

$a_{10} - 2a_8 = 6a_9$

$$\frac{a_{10} - 2a_8}{2a_9} = \frac{6}{2} = 3$$

JEE Main 2024

If α, β are the roots of the equation, $x^2 - x - 1 = 0$ and $S_n = 2023\alpha^n + 2024\beta^n$, then:

(A) $2S_{12} = S_{11} + S_{10}$

(B) $S_{12} = S_{11} + S_{10}$

(C) $2S_{11} = S_{12} + S_{10}$

(D) $S_{11} = S_{10} + S_{12}$

① $1x^2 - 1x - 1 = 0$ → α, β

② $S_n = 2023\alpha^n + 2024\beta^n$

$1 S_n - 1 S_{n-1} - 1 S_{n-2} = 0$

$$S_n - S_{n-1} - S_{n-2} = 0$$

$n = 12$

$S_{12} - S_{11} - S_{10} = 0 \Rightarrow S_{12} = S_{10} + S_{11}$

JEE Main 2024

Let α, β be roots of $x^2 + \sqrt{2}x - 8 = 0$. If $U_n = \alpha^n + \beta^n$, then $\frac{U_{10}+\sqrt{2}U_9}{2U_8}$ is equal to ________.

① $x^2 + \sqrt{2}x - 8 = 0$ → α, β

② $U_n = \alpha^2 + \beta^n$

$U_n + \sqrt{2}\, U_{n-1} - 8\, U_{n-2} = 0$

$n = 10$:

$U_{10} + \sqrt{2}\, U_9 - 8\, U_8 = 0$

$U_{10} + \sqrt{2}\, U_9 = 8\, U_8$

Ans: 4

JEE Main 2024

Let $\alpha, \beta; \alpha > \beta$, be the roots of the equation $x^2 - \sqrt{2}x - \sqrt{3} = 0$. Let $P_n = \alpha^n - \beta^n, n \in \mathbb{N}$. Then $(11\sqrt{3} - 10\sqrt{2})P_{10} + (11\sqrt{2} + 10)P_{11} - 11P_{12}$ is equal to:

(A) $10\sqrt{3}P_9$ (B) $11\sqrt{3}P_9$ (C) $10\sqrt{2}P_9$ (D) $11\sqrt{2}P_9$

① $x^2 - \sqrt{2}x - \sqrt{3} = 0$ → α, β

$P_n = \alpha^n - \beta^n \quad n \geq 1$

$P_n - \sqrt{2}P_{n-1} - \sqrt{3}P_{n-2} = 0$

$n = 12$: $P_{12} - \sqrt{2}P_{11} - \sqrt{3}P_{10} = 0$

$n = 11$: $P_{11} - \sqrt{2}P_{10} - \sqrt{3}P_9 = 0$

$(11\sqrt{3} - 10\sqrt{2})P_{10} + (11\sqrt{2} + 10)P_{11} - 11P_{12}$

$-11(P_{12} - \sqrt{2}P_{11} - \sqrt{3}P_{10}) + 10(P_{11} - \sqrt{2}P_{10})$

zero $\sqrt{3}P_9$

Ans: $= 10\sqrt{3}P_9$

JEE Main 2022 Ans: 16

Let $\alpha, \beta (\alpha > \beta)$ be the roots of the quadratic equation $x^2 - x - 4 = 0$. If $P_n = \alpha^n - \beta^n, n \in \mathbb{N}$, then

$$\frac{P_{15}P_{16} - P_{14}P_{16} - P_{15}^2 + P_{14}P_{15}}{P_{13}P_{14}}$$

is equal to 16.

① $x^2 - x - 4 = 0$ → α, β

② $P_n = \alpha^n - \beta^n$

$P_n - P_{n-1} - 4P_{n-2} = 0$

$n = 15$: $P_{15} - P_{14} = 4P_{13}$

$n = 16 \Rightarrow$

$$\frac{P_{16}(P_{15} - P_{14}) - P_{15}(P_{15} - P_{14})}{P_{13}P_{14}}$$

$$\left(\frac{P_{15} - P_{14}}{P_{13}}\right)\left(\frac{P_{16} - P_{15}}{P_{14}}\right)$$

$(4)(4) = 16$

JEE Main 2022

For a natural number n, let $\alpha_n = 19^n - 12^n$. Then, the value of

$$\frac{31\alpha_9 - \alpha_{10}}{57\alpha_8} = \frac{31 S_9 - S_{10}}{57 S_8}$$

is ________.

$\alpha_n = 19^n - 12^n$

$S_n = \alpha^n \pm \beta^n$

$\alpha = 19 \quad \beta = 12$

$x^2 - (\alpha+\beta)x + \alpha\beta = 0$

$x^2 - (19+12)x + 19.12 = 0$

$x^2 - 31x + 228 = 0$ → α, β

$S_n = \alpha^n - \beta^n$

$S_n - 31S_{n-1} + 228S_{n-2} = 0$

$n = 10$

$S_{10} - 31S_9 + 228S_8 = 0$

$228 S_8 = 31S_9 - S_{10}$

Ans: 4

JEE Main 2025 (3 Apr Shift 1) Ans:(C)

Let α and β be the roots of $x^2 + \sqrt{3}x - 16 = 0$, and γ and δ be the roots of $x^2 + 3x - 1 = 0$. If $P_n = \alpha^n + \beta^n$ and $Q_n = \gamma^n + \delta^n$, then

$$\frac{P_{25} + \sqrt{3}P_{24}}{2P_{23}} + \frac{Q_{25} - Q_{23}}{Q_{24}}$$

is equal to:

(A) 3 (B) 4 (C) 5 (D) 7

JEE Main 2025 (2 Apr Shift 1) Ans:(B)

Let $P_n = \alpha^n + \beta^n, n \in \mathbb{N}$. If $P_{10} = 123, P_9 = 76, P_8 = 47$ and $P_1 = 1$, then the quadratic equation having roots $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ is:

(A) $x^2 - x + 1 = 0$

(B) $x^2 + x - 1 = 0$

(C) $x^2 - x - 1 = 0$

(D) $x^2 + x + 1 = 0$

Newton's Method: Type-2 Identities

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JEE Main 2024 (09 Apr Shift 1)

Let α, β be the roots of the equation $x^2 + 2\sqrt{2}x - 1 = 0$. The quadratic equation, whose roots are $\alpha^4 + \beta^4$ and $\frac{1}{10}(\alpha^6 + \beta^6)$, is:

(A) $x^2 - 190x + 9466 = 0$

(B) $x^2 - 180x + 9506 = 0$

(C) $x^2 - 195x + 9506 = 0$

(D) $x^2 - 195x + 9466 = 0$

JEE Main 2023 (25 Jan Shift 2) Ans: 45

Let $a \in \mathbb{R}$ and let α, β be the roots of the equation

$$x^2 + 60^{1/4}x + a = 0.$$

If $\alpha^4 + \beta^4 = -30$, then the product of all possible values of a is ________.

Quadratic Equation: Section 2

Newton's Method: Type-3 Complex Roots

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JEE Main 2023 (13 Apr Shift 2)

Let α, β be the roots of the equation $x^2 - \sqrt{2}x + 2 = 0$. Then $\alpha^{14} + \beta^{14}$ is equal to:

(A) -64 (B) $-64\sqrt{2}$ (C) -128 (D) $-128\sqrt{2}$

Solution: JEE Main 2023 (13 Apr Shift 2)

Given $x^2 - \sqrt{2}x + 2 = 0$ with roots α, β.
To find: $\alpha^{14} + \beta^{14}$.

Let $S_n = \alpha^n + \beta^n$. Our goal is to find S_{14}.

Step 1: Apply Newton's Sums Formula

$$S_n - \sqrt{2}S_{n-1} + 2S_{n-2} = 0$$

Applying this for $n = 14$ and $n = 13$:

$$S_{14} - \sqrt{2}S_{13} + 2S_{12} = 0 \quad (1)$$

$$S_{13} - \sqrt{2}S_{12} + 2S_{11} = 0 \quad (2)$$

Step 2: Eliminate terms to find a pattern
Multiply equation (2) by $\sqrt{2}$:

$$\sqrt{2}S_{13} - 2S_{12} + 2\sqrt{2}S_{11} = 0 \quad (3)$$

Now, add equation (1) and equation (3):

$$(S_{14} - \sqrt{2}S_{13} + 2S_{12}) + (\sqrt{2}S_{13} - 2S_{12} + 2\sqrt{2}S_{11}) = 0$$

$$S_{14} + 2\sqrt{2}S_{11} = 0$$

This reveals a pattern. By repeating this process, we can see that:

$$S_n + 2\sqrt{2}S_{n-3} = 0 \implies S_n = -2\sqrt{2}S_{n-3}$$

So, we can establish a chain of relations:

$$S_{14} = -2\sqrt{2}S_{11}$$

$$S_{11} = -2\sqrt{2}S_8$$

$$S_8 = -2\sqrt{2}S_5$$

$$S_5 = -2\sqrt{2}S_2$$

Step 3: Calculate the base case, S_2

$$\alpha + \beta = \sqrt{2}$$

$$\alpha\beta = 2$$

$$S_2 = (\alpha + \beta)^2 - 2\alpha\beta = (\sqrt{2})^2 - 2(2)$$

$$= 2 - 4 = -2$$

Solution: JEE Main 2023 (13 Apr Shift 2)

Step 4: Work backwards to find S_{14}

$$S_5 = -2\sqrt{2}S_2 = -2\sqrt{2}(-2) = 4\sqrt{2}$$

$$S_8 = -2\sqrt{2}S_5 = -2\sqrt{2}(4\sqrt{2}) = -16$$

$$S_{11} = -2\sqrt{2}S_8 = -2\sqrt{2}(-16) = 32\sqrt{2}$$

$$S_{14} = -2\sqrt{2}S_{11} = -2\sqrt{2}(32\sqrt{2}) = -64(2) = -128$$

Thus, the value of $\alpha^{14} + \beta^{14}$ is -128.
The correct option is (C).

JEE Main 2023 (12 Apr Shift 1)

Let α, β be the roots of the quadratic equation $x^2 + \sqrt{6}x + 3 = 0$. Then

$$\frac{\alpha^{23} + \beta^{23} + \alpha^{14} + \beta^{14}}{\alpha^{15} + \beta^{15} + \alpha^{10} + \beta^{10}}$$

is equal to:

(A) 81 (B) 9 (C) 72 (D) 729

Solution: JEE Main 2023 (12 Apr Shift 1)

Given the quadratic equation $x^2 + \sqrt{6}x + 3 = 0$ with roots α, β. We need to find the value of the expression:

$$E = \frac{\alpha^{23} + \beta^{23} + \alpha^{14} + \beta^{14}}{\alpha^{15} + \beta^{15} + \alpha^{10} + \beta^{10}}$$
$$= \frac{S_{23} + S_{14}}{S_{15} + S_{10}}$$

Step 1: Find a relation for higher powers of the roots.

Since α is a root, it satisfies the equation: $\alpha^2 + \sqrt{6}\alpha + 3 = 0.$

shifting middle term: $\alpha^2 + 3 = -\sqrt{6}\alpha.$

Squaring both sides:

$$(\alpha^2 + 3)^2 = (-\sqrt{6}\alpha)^2$$
$$\alpha^4 + 6\alpha^2 + 9 = 6\alpha^2$$
$$\alpha^4 + 9 = 0 \implies \alpha^4 = -9$$

Similarly, for the root β, we have $\beta^4 = -9$.

Step 2: Find the value of $\alpha^2 + \beta^2$.

Sum of roots: $\alpha + \beta = -\sqrt{6}$

Product of roots: $\alpha\beta = 3$

Now, we find $\alpha^2 + \beta^2$:

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 6 - 6 = 0$$

Solution: JEE Main 2023 (12 Apr Shift 1)

Step 3: Simplify the terms in the expression.
Let $S_n = \alpha^n + \beta^n$. Since $S_2 = 0$:

$$S_{14} = (\alpha^4)^3 S_2 = (-9)^3(0) = 0.$$

$$S_{10} = (\alpha^4)^2 S_2 = (81)(0) = 0.$$

Step 4: Substitute and solve.
The expression simplifies to:

$$E = \frac{S_{23} + 0}{S_{15} + 0} = \frac{\alpha^{23} + \beta^{23}}{\alpha^{15} + \beta^{15}}$$

We express the numerator and denominator in terms of lower powers:

$$\text{Numerator: } S_{23} = (\alpha^4)^5 S_3 = (-9)^5 S_3.$$

$$\text{Denominator: } S_{15} = (\alpha^4)^3 S_3 = (-9)^3 S_3.$$

So, the final expression is:

$$E = \frac{(-9)^5 S_3}{(-9)^3 S_3} = (-9)^2 = 81$$

The correct option is (A).

Newton's Method: Type-4
Shift Middle Term

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JEE Adv. 2011

Let α and β be the roots of $x^2 - 6x - 2 = 0$ with $\alpha > \beta$. If $a_n = \alpha^n - \beta^n$ for $n \geq 1$, then the value of

$$\frac{a_{10} - 2a_8}{2a_9}$$

is ________.

JEE Main 2023 (11 Apr Shift 1) Ans: 51

If a and b are the roots of the equation

$$x^2 - 7x - 1 = 0,$$

then the value of $\frac{a^{21}+b^{21}+a^{17}+b^{17}}{a^{19}+b^{19}}$ is equal to ______.

Solution: JEE Main 2023 (11 Apr Shift 1)

Equation $x^2 - 7x - 1 = 0$, with roots a and b.
We need to find the value of the expression:

$$E = \frac{a^{21} + b^{21} + a^{17} + b^{17}}{a^{19} + b^{19}}$$

Step 1: Shift the middle term and square

Since a is a root of the equation, it must satisfy it.

$$a^2 - 7a - 1 = 0$$
$$a^2 - 1 = 7a$$
$$(a^2 - 1)^2 = (7a)^2$$
$$a^4 - 2a^2 + 1 = 49a^2$$
$$a^4 + 1 = 51a^2$$

Also, $b^4 + 1 = 51b^2$

Step 2: Simplify the required expression.

Let's rearrange the numerator of the expression E:

$$\begin{aligned} &a^{21} + b^{21} + a^{17} + b^{17} \\ &= a^{17}(a^4 + 1) + b^{17}(b^4 + 1) \\ &= a^{17}(51a^2) + b^{17}(51b^2) \\ &= 51a^{19} + 51b^{19} \\ &= 51(a^{19} + b^{19}) \end{aligned}$$

Step 3: Calculate the final value of E.

$$E = \frac{51(a^{19} + b^{19})}{a^{19} + b^{19}} = 51$$

JEE Main 2021 — Ans: 50

Let α, β be two roots of the equation

$$x^2 + 20^{1/4}x + 5^{1/2} = 0$$

Then $\alpha^8 + \beta^8$ is equal to:

Solution: JEE Main 2021 (Method-1)

Given the equation $x^2 + 20^{1/4}x + 5^{1/2} = 0$ with roots α, β. We need to find the value of $\alpha^8 + \beta^8$.

Step 1: Find the sum and product of the roots.

Sum of roots: $\alpha + \beta = -20^{1/4}$

Product of roots: $\alpha\beta = 5^{1/2} = \sqrt{5}$

Step 2: Calculate $\alpha^2 + \beta^2$.

We use the identity $(\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$.

$$
\begin{aligned}
\alpha^2 + \beta^2 &= (\alpha + \beta)^2 - 2\alpha\beta \\
&= (-20^{1/4})^2 - 2(\sqrt{5}) \\
&= 20^{1/2} - 2\sqrt{5} \\
&= \sqrt{20} - 2\sqrt{5} \\
&= 2\sqrt{5} - 2\sqrt{5} = 0
\end{aligned}
$$

Step 3: Calculate $\alpha^4 + \beta^4$.

Now we square the result from Step 2:

$$
\begin{aligned}
\alpha^4 + \beta^4 &= (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2 \\
&= (0)^2 - 2(\alpha\beta)^2 \\
&= -2(\sqrt{5})^2 \\
&= -2(5) = -10
\end{aligned}
$$

Step 4: Calculate the final value $\alpha^8 + \beta^8$.

Finally, we square the result from Step 3:

$$
\begin{aligned}
\alpha^8 + \beta^8 &= (\alpha^4 + \beta^4)^2 - 2\alpha^4\beta^4 \\
&= (-10)^2 - 2(\alpha\beta)^4 \\
&= 100 - 2(\sqrt{5})^4 \\
&= 100 - 2(5^2) \\
&= 100 - 2(25) \\
&= 100 - 50 = 50
\end{aligned}
$$

Solution: JEE Main 2021 (Method-2)

Given the equation $x^2 + 20^{1/4}x + 5^{1/2} = 0$.

Step 1: Rearrange the Equation

For a root x of the equation, we can write:

$$x^2 + (5)^{1/2} = -x(20)^{1/4}$$
$$x^2 + \sqrt{5} = -x \cdot (20)^{1/4}$$

Step 2: Square Both Sides

$$(x^2 + \sqrt{5})^2 = \left(-x \cdot (20)^{1/4}\right)^2$$
$$(x^2)^2 + 2(x^2)(\sqrt{5}) + (\sqrt{5})^2 = x^2 \cdot (20)^{1/2}$$
$$x^4 + 2\sqrt{5}x^2 + 5 = x^2 \cdot \sqrt{20}$$
$$x^4 + 2\sqrt{5}x^2 + 5 = x^2 \cdot 2\sqrt{5}$$

The term $2\sqrt{5}x^2$ cancels from both sides:

$$x^4 + 5 = 0$$
$$x^4 = -5$$

This derived equation must be satisfied by both roots, α and β, of the original quadratic equation.

Step 3: Calculate the Final Value

Since α and β satisfy $x^4 = -5$, we have:

$$\alpha^4 = -5$$
$$\beta^4 = -5$$

Now, we find the 8th power by squaring these results:

$$\alpha^8 = (\alpha^4)^2 = (-5)^2 = 25$$
$$\beta^8 = (\beta^4)^2 = (-5)^2 = 25$$

Finally, we find the required sum:

$$\alpha^8 + \beta^8 = 25 + 25 = 50$$

The answer is 50.

JEE Main 2021 Ans: (C)

If α and β are distinct roots of the equation

$$x^2 + 3^{1/4}x + 3^{1/2} = 0,$$

then the value of

$$\alpha^{96}(\alpha^{12} - 1) + \beta^{96}(\beta^{12} - 1)$$

is equal to:

(A) $56 \cdot 3^{25}$ (B) $56 \cdot 3^{24}$ (C) $52 \cdot 3^{24}$ (D) $28 \cdot 3^{25}$

Solution: JEE Main 2021

The given equation is $x^2 + 3^{1/4}x + 3^{1/2} = 0$.
To find: $\alpha^{96}(\alpha^{12} - 1) + \beta^{96}(\beta^{12} - 1)$.

Step 1: Shifting middle term and squaring.
Since α is a root, it satisfies the equation:

$$\alpha^2 + 3^{1/4}\alpha + \sqrt{3} = 0$$
$$\alpha^2 + \sqrt{3} = -3^{1/4}\alpha$$
$$(\alpha^2 + \sqrt{3})^2 = (-3^{1/4}\alpha)^2$$
$$\alpha^4 + 2\sqrt{3}\alpha^2 + 3 = \sqrt{3}\alpha^2$$

Simplifying this expression gives:

$$\alpha^4 + 3 = -\sqrt{3}\alpha^2$$

Step 2: Continue simplifying to find α^{12}.
Squaring the equation $\alpha^4 + 3 = -\sqrt{3}\alpha^2$:

$$(\alpha^4 + 3)^2 = (-\sqrt{3}\alpha^2)^2$$
$$\alpha^8 + 6\alpha^4 + 9 = 3\alpha^4$$

Rearranging the terms to find a value for α^8:

$$\alpha^8 = -3\alpha^4 - 9$$

Now, to find α^{12}, we can multiply the equation $\alpha^8 = -3\alpha^4 - 9$ by α^4:

$$\alpha^{12} = -3\alpha^8 - 9\alpha^4$$

Substitute the expression for α^8 back into this equation:

$$\alpha^{12} = -3(-3\alpha^4 - 9) - 9\alpha^4$$
$$\alpha^{12} = 9\alpha^4 + 27 - 9\alpha^4$$
$$\alpha^{12} = 27$$
$$\beta^{12} = 27$$

Solution: JEE Main 2021

Step 3: Calculate the required expression.

$$\alpha^{96}(\alpha^{12}-1)+\beta^{96}(\beta^{12}-1)$$

First, let's find the value of α^{96} and β^{96}:

$$\alpha^{96}=(\alpha^{12})^8=(27)^8=(3^3)^8=3^{24}$$
$$\beta^{96}=(\beta^{12})^8=(27)^8=3^{24}$$

Now, substitute these values back into the expression:

$$\begin{aligned}&\alpha^{96}(\alpha^{12}-1)+\beta^{96}(\beta^{12}-1)\\&=3^{24}(27-1)+3^{24}(27-1)\\&=3^{24}(26)+3^{24}(26)\\&=2\times 26\times 3^{24}\\&=52\cdot 3^{24}\end{aligned}$$

Thus, the correct option is (C).