Basic Mathematics: Section 4

Polynomials

mathbyiiserite

Definition of a Polynomial

An algebraic expression of the form:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x + a_0$$

where:

- $a_n, a_{n-1}, \ldots, a_0$ are constants called **coefficients**.
- The powers of x must be **whole numbers**
- $a_n \neq 0$.

- $a_n x^n$ is the **leading term**.
- a_n is the **leading coefficient**.
- n (the highest power) is the **degree** of the polynomial.

Division of Polynomials

If $P(x)$ **(dividend)** and $d(x)$ **(divisor)** are two polynomials such that the degree of $d(x)$ is less than or equal to the degree of $P(x)$, and $d(x) \neq 0$, then there exist unique polynomials $Q(x)$ **(quotient)** and $R(x)$ **(remainder)** such that:

$$P(x) = d(x) \times Q(x) + R(x)$$

where the degree of the remainder $R(x)$ is strictly less than the degree of the divisor $d(x)$,

Analogy with numbers: $49 \div 6 \implies 49 = 6 \times 8 + 1$

$$\boxed{\textit{Dividend} = \textit{Divisor} \times \textit{Quotient} + \textit{Remainder}}$$

Long Division Method

$p(-2) = -72 + 76 - 38 - 6 = -40$

Question:1

Divide $P(x) = 9x^3 + 19x^2 + 19x - 6$ by $d(x) = x + 2$.

$x - a \longrightarrow p(a) = \text{Remainder}$

$x + 2 = 0 \Rightarrow x = -2 \Rightarrow p(-2)$

$$
\begin{array}{r|l}
 & 9x^2 + x + 17 \\
\hline
x + 2 & 9x^3 + 19x^2 + 19x - 6 \\
 & 9x^3 + 18x^2 \\
\hline
 & 0 + x^2 + 19x - 6 \\
 & x^2 + 2x \\
\hline
 & 0 + 17x - 6 \\
 & 17x + 34 \\
\hline
 & -40
\end{array}
$$

$9x^3 + 19x^2 + 19x - 6 = (x+2)(9x^2 + x + 17) + (-40)$

put $x = -2$

$= -40$

Question:2

$P(4) = 256 + 128 - 208 + 24 =$

Divide the polynomial $x^4 + 2x^3 - 13x^2 + 24$ by the linear factor $(x - 4)$.

$$x^3 + 6x^2 + 11x + 44$$

$0x$

$$x - 4 \,\big)\, x^4 + 2x^3 - 13x^2 + 24$$

$x^4 - 4x^3$

$6x^3 - 13x^2 + 24$

$6x^3 - 24x^2$

$0 + 11x^2 + 24$

$11x^2 - 44x$

$0 + 44x + 24$

$44x - 176$

200

$x - 4 = 0$

$x = 4$

$13 \times 16 = 208$

4

$128 + 48 + 24 = 200$

Question:3

Ans: 1

If $x = 2 - \sqrt{3}$, then find the value of $x^3 - x^2 - 11x + 4$

$(2-\sqrt{3})^3 - (2-\sqrt{3})^2 - 11(2-\sqrt{3}) + 4$

M-2

$\boxed{x = 2-\sqrt{3}}$

$x - 2 = -\sqrt{3}$

Square

$x^2 - 4x + 4 = 3$

$\boxed{x^2 - 4x + 1 = 0}$

$$\begin{array}{r l}
 & x + 3 \\
x^2 - 4x + 1 \,\big) & x^3 - x^2 - 11x + 4 \\
 & x^3 - 4x^2 + x \\
 & 3x^2 - 12x + 4 \\
 & 3x^2 - 12x + 3 \\
 & 0 + 0 + 1
\end{array}$$

$(x^3 - x^2 - 11x + 4) = (x^2 - 4x + 1)(x + 3) + 1$

0

Question:4

Find the quotient and remainder when the polynomial $2x^3 + 9x^2 + 10x + 3$ is divided by $x^2 + 1$.

$$2x + 9$$

$$x^2 + 0x + 1 \,\big)\, 2x^3 + 9x^2 + 10x + 3$$

$$2x^3 + 0x^2 + 2x$$

$$0 + 9x^2 + 8x + 3$$

$$9x^2 + 0x + 9$$

$$0 + 8x - 6$$

Remainder Theorem

The Remainder Theorem

If $P(x)$ is divided by $(x - a)$, then remainder is $P(a)$.

$$x - a \,\overline{)\; P(x) \;}$$

$$x - a = 0$$

$$x = a$$

$$P(a)$$

Question:1

Find the remainder when $P(x) = x^3 - 7x^2 + 6x + 9$ is divided by $(x - 6)$.

$x - 6 = 0$

$x = 6$

Remainder $= P(6)$

$$\begin{array}{r} 36 \\ \times\; 7 \\ \hline 252 \end{array}$$

4

$= 216 - 252 + 36 + 9$

$= \cancel{252} + 9 - \cancel{252}$

$= 9$ //

Question:2

Find the remainder when $P(x) = x^3 - 10x^2 + 6x + 10$ is divided by $(x + \frac{1}{2})$.

$$x + \frac{1}{2} = 0$$

$$x = \frac{-1}{2}$$

$$\text{Remainder} = P\left(-\frac{1}{2}\right)$$

$$P(x) \longrightarrow (x-a) \longrightarrow P(a)$$

Factor Theorem

$f(x) = (x+2)(x-\alpha)(x-\beta) + 3$

The Factor Theorem

Let $P(x)$ be a polynomial and let a be any real number.

$f(-2) = 3$

- If $P(a) = 0$, then $(x - a)$ is a factor of $P(x)$.
- Conversely, if $(x - a)$ is a factor of $P(x)$, then $P(a) = 0$.

eg $P(x) = x^2 - 5x + 6 = (x-2)(x-3)$

$P(1) = 1^2 - 5 + 6 = 2$

$P(2) = 4 - 10 + 6 = 0$

$P(3) = 9 - 15 + 6 = 0$

6

-2 -3

HAD

$f(x)$ be cubic poly $f(2) = 0$

$f(3) = 0$ and $\boxed{f'(2) = 1}$ then $\int_1^2 f(x)\,dx$

$f(x) = (x-2)(x-3)(x-a)$

$f'(2) = 1$

Question:1

Check whether $(x-1)$ is a factor of $F(x) = 2\sqrt{2}x^3 + 5\sqrt{2}x^2 - 7\sqrt{2}$.

$$2\sqrt{2}x^3 + 5\sqrt{2}x^2 - 7\sqrt{2} = (x-1)(\text{Quad})$$

$$F(1) = \underline{2\sqrt{2} + \cancel{5}\sqrt{2}} - \cancel{7}\sqrt{2} = 0$$

Question:2

Show that $(x - 3)$ is a factor of $F(x) = x^3 - 3x^2 + 4x - 12$.

$\underset{0}{\parallel}$

$F(3) = 0$

$f(3) = 27 - 27 + 12 - 12$

$= 0$ //

Question:3

Prove that $x^2 - 3x + 2$ is a factor of $F(x) = 2x^4 - 6x^3 + 3x^2 + 3x - 2$.

Quad

Bi-quad

2

-2 -1

$F(x) = (\text{Quad})(\text{Quad})'$

$2x^4 - 6x^3 + 3x^2 + 3x - 2 = (x-1)(x-2)(\text{Quad})'$

$F(1) = 0$

$F(2) = 0$

Expression vs Function vs Equation

Distinguishing the terms:

- **Polynomial Expression:** $3x^2 - 5x + 6.$
- **Polynomial Function:** $f(x) = 3x^2 - 5x + 6.$
- **Polynomial Equation:** $3x^2 - 5x + 6 = 0.$

Zeroes vs Roots

Note:

- **Zeroes:** Values of x for which the value of a *polynomial function* $P(x)$ becomes zero. That is, $P(x) = 0$.
- **Roots:** Values of x which satisfy a *polynomial equation*.

- For the function $f(x) = x^2 - 5x + 6$:
  $x = 2$ and $x = 3$ are the **zeroes** of the function $f(x)$.
- For the equation $x^2 - 5x + 6 = 0$:
  $x = 2$ and $x = 3$ are the **roots** of the equation $x^2 - 5x + 6 = 0$.

Graphical Meaning of Zeroes/Roots

Note:

Graphically, the real zeroes or roots correspond to the **x-coordinates** of the points where the graph of $y = P(x)$ intersects or touches the **x-axis**.

Example: For $f(x) = x^2 - 7x + 6 = (x - 1)(x - 6)$

Question:1

Given that $x = 1$ is a zero of the polynomial $f(x) = x^3 - 2x^2 - 5x + 6$, find the other two zeros.

(A) 2, 3 (B) −2, 3 (C) 2, −3 (D) −2, −3

$x = 1$ is zero of $f(x) \Rightarrow f(1) = 0$

$\Rightarrow (x-1)$ is factor of $F(x)$

$$x^3 - 2x^2 - 5x + 6 = x^2(x-1) - x(x-1) - 6(x-1)$$

$$-x^2 - x^2 + x - 6x + 6$$

$$= (x-1)(x^2 - x - 6)$$

$$= (x-1)(x-3)(x+2)$$

-6

$-3 \quad +2$

$x = 3, \; x = -2$

Question:2

$P(-1)=0$ $P(2)=0$

If $x = -1$ and $x = 2$ are zeros of $P(x) = x^4 - x^3 - 7x^2 + x + 6$, find the remaining zeros.

(A) $1, -3$ (B) $-1, 3$ (C) $1, 3$ (D) $-1, -3$

$(x=-1)$ is zero $\Rightarrow$ $(x+1)$ is factor

$x=2$ is zero $\Rightarrow$ $(x-2)$ is factor.

$P(x) = x^4 - x^3 - 7x^2 + x + 6 = (x+1)(x-2)\,(\text{Quad})$

$= (x^2 - x - 2)\,(\text{Quad})$

$$
\begin{array}{r|l}
 & x^2 - 5 \\
\hline
x^2 - x - 2 & x^4 - x^3 - 7x^2 + x + 6 \\
 & x^4 - x^3 - 2x^2 \\
\hline
 & \quad -5x^2 + x + 6 \\
 & \quad -5x^2 + 5x + 10 \\
\hline
 & \quad\quad -4x - 4
\end{array}
$$

Question:3

e.g. $f(x)$ is quad having $1, -2$ roots. $f(x) = k(x-1)(x+2)$

The polynomial $F(x) = 3x^3 + ax^2 + bx + c$ has the roots 1, -2, and 3. Find the coefficients a, b, and c.

Roots are 1, -2, and 3

$F(x) = 3x^3 + ax^2 + bx + c = k(x-1)(x+2)(x-3)$

$k = 3$

$3x^3 + ax^2 + bx + c = 3\left[(x^2 + x - 2)(x-3)\right]$

$3x^3 + ax^2 + bx + c = 3x^3 - 6x^2 - 15x + 18$

$a = -6$
$b = -15$
$c = 18$