Factorization

mathbyiiserite

$p(a) = 0 \Rightarrow (x-a)$

$(x-a)$ if factor $\Rightarrow p(a) = 0$

Algebraic Identities

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1. $(a+b)^2 = a^2 + 2ab + b^2$

2. $(a-b)^2 = a^2 - 2ab + b^2$

3. $a^2 + b^2 = (a-b)^2 + 2ab$
   $a^2 + b^2 = (a+b)^2 - 2ab$

4. $a^2 - b^2 = (a-b)(a+b)$

5. $(a+b)^2 = (a-b)^2 + 4ab$
   $(a-b)^2 = (a+b)^2 - 4ab$

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6. $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

7. $a^3 + b^3 = (a+b)^3 - 3ab(a+b)$

$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

8. $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

9. $a^3 - b^3 = (a-b)^3 + 3ab(a-b)$

$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

$a^3 + b^3 = (a+b)^3 - 3a^2b - 3ab^2$

$\boxed{a^3 + b^3 = (a+b)^3 - 3ab(a+b)}$

$a^3 + b^3 = (a+b)((a+b)^2 - 3ab)$

$\boxed{a^3 + b^3 = (a+b)(a^2 - ab + b^2)}$

9.

$\boxed{a^3 - b^3 = (a-b)^3 + 3ab(a-b)}$

$= (a-b)[(a-b)^2 + 3ab]$

$\boxed{a^3 - b^3 = (a-b)(a^2 + ab + b^2)}$

$(a+(-b))^3 = a^3 + 3a^2(-b) + 3a(-b)^2 + (-b)^3$

8. $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

$a^3 - b^3 = (a-b)^3 + 3a^2b - 3ab^2$

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10. $(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca$

11. $\boxed{a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)}$

$$a^3+b^3+c^3-3abc = \frac{1}{2}(a+b+c)\left[(a-b)^2+(b-c)^2+(c-a)^2\right]$$

* If $\boxed{a+b+c=0}$ or $\boxed{a=b=c,}$ then $a^3+b^3+c^3 = 3abc$

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12. $a^4+a^2+1 = (a^2+1)^2-a^2 = (a^2+a+1)(a^2-a+1)$

10. $(a+b+c)^2 = (a+b)^2 + (c)^2 + 2(a+b)(c)$

$= a^2+b^2+c^2+2ab+2ac+2bc$

* $(a+b+c+d)^2 = a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd$

Question 1

If $x - y = 2$ and $xy = 24$, find the value of $\frac{1}{x} + \frac{1}{y}$. $= \frac{x+y}{xy}$

(A) $\frac{1}{12}$ (B) $\frac{5}{12}$ (C) $\pm\frac{5}{12}$ (D) $\frac{1}{24}$

$x - y = 2$ $xy = 24$

$(a-b)^2 = (a+b)^2 - 4ab$

$(x-y)^2 = (x+y)^2 - 4xy$

$(2)^2 = (x+y)^2 - 4(24)$

$x + y = \pm 10$

$= \frac{\pm 10}{24}$

$= \frac{\pm 5}{12}$

Question 2

If $x^3 + y^3 = 7$ and $xy(x + y) = -2$, find the value of $x^2 + y^2$.

(A) 1 (B) 3 (C) 5 (D) 9

$$a^3 + b^3 = (a+b)^3 - 3ab(a+b)$$

$$x^3 + y^3 = (x+y)^3 - 3xy(x+y)$$

$$7 = (x+y)^3 - 3(-2)$$

$$(x+y)^3 = 1$$

$$\boxed{x+y=1}$$

$$xy(x+y) = -2$$

$$xy(1) = -2$$

$$\boxed{xy=-2}$$

$$\text{Req} = x^2+y^2 = (x+y)^2 - 2xy$$

$$= 1^2 - 2(-2)$$

$$= 5$$

Question 3

Given that $x^4 + y^4 = 82$, $xy = 3$ and $x, y > 0$, find the values of x and y.

(A) $\{1, 2\}$ (B) $\{1, 3\}$ (C) $\{2, 3\}$ (D) $\{3, 4\}$

$$x^4 + y^4 = \boxed{x^2}^2 + (y^2)^2$$

$$a^2 + b^2 = (a+b)^2 - 2ab$$

$$\square^2 + \bigcirc^2 = (\square + \bigcirc)^2 - 2\square\bigcirc$$

$$82 = (x^2+y^2)^2 - 2x^2y^2$$

$$82 = \left[(x+y)^2 - 2xy\right]^2 - 2(xy)^2$$

$$82 = \left[(x+y)^2 - 6\right]^2 - 2(9)$$

$$(x+y)^2 - 6 = 10 \quad \text{OR} \quad (x+y)^2 - 6 = -10$$

Reject

$$x+y = \pm 4$$

$$\boxed{x+y=4} \ \& \ \boxed{xy=3}$$

$$x^2 - (\alpha+\beta)x + \alpha\beta = 0$$

α, β

$$x^2 - 4x + 3 = 0$$

3

-3 -1

$x = 3$ $x = 1$

Question.4

If $x + y = 3$ and $xy = 2$, then the value of $x^3 - y^3$ is:

(A) 6 (B) 7 (C) 8 (D) 0

$$(x-y)^2 = (x+y)^2 - 4xy$$
$$= (3)^2 - 4(2)$$
$$(x-y)^2 = 1$$
$$x-y = \pm 1$$

$$x^3 - y^3 = (x-y)^3 + 3xy(x-y)$$
$$= (1)^3 + 3(2)(1)$$
$$= 7$$

$$(-1)^3 + 3(2)(-1)$$
$$-1-6$$
$$-7$$

Question.5

If $x - \frac{1}{x} = 5$, find the value of $x^3 - \frac{1}{x^3}$.

(A) 110 (B) 125 (C) 140 (D) 155

$$(x)^3 - \left(\frac{1}{x}\right)^3 = \left(x - \frac{1}{x}\right)^3 + 3\cancel{(x)}\left(\frac{1}{\cancel{x}}\right)\left(x - \frac{1}{x}\right)$$

$$a^3 - b^3 = (a-b)^3 + 3ab(a-b)$$

$$= (5)^3 + 3(5)$$

$$= 125 + 15$$

$$= 140$$

Question.6

Let p, q be real numbers satisfying $p^2 - q^2 = 4$ and $2pq = 3$. Then, the value of $(p^2 + q^2)^2$ is equal to:

$a - b = 4$ $\qquad$ $2.\sqrt{a}\sqrt{b} = 3$

(A) 1 $\qquad$ (B) 9 $\qquad$ (C) 16 $\qquad$ $4ab = 9$ $\qquad$ (D) 5

$(p-q)(p+q) = 4$ $\qquad$ $ab = \frac{9}{4}$

$(a+b)^2 = (a-b)^2 + 4ab$

$(p^2+q^2)^2 = (p^2-q^2)^2 + 4\,\underline{p^2q^2}$

$= (4)^2 + 4\left(\frac{3}{2}\right)^2$

$= 16 + \cancel{4} \times \frac{9}{\cancel{4}}$

$= 25$ //

Question.7 JEE Main 2021

If $a + b + c = 1$, $ab + bc + ca = 2$ and $abc = 3$, then the value of $a^4 + b^4 + c^4$ is:

(A) -7 (B) 1 (C) 13 (D) 23

$(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ac)$ —— ⊛

$(1)^2 = a^2+b^2+c^2+2(2)$

$\therefore \boxed{a^2+b^2+c^2=-3}$

$ab+bc+ac=2$ squaring

$4 = (ab)^2+(bc)^2+(ac)^2+2(ab^2c+abc^2+a^2bc)$

$4 = (ab)^2+(bc)^2+(ac)^2+2\cancel{abc}[\cancel{b+c+a}]$

$3 \qquad 1$

$(ab)^2+(bc)^2+(ac)^2=-2$

put $a \to a^2$
$b \to b^2$
$c \to c^2$

$(a^2+b^2+c^2)^2 = a^4+b^4+c^4+2(a^2b^2+b^2c^2+a^2c^2)$

$(-3)^2 = a^4+b^4+c^4+2(-2) \Rightarrow a^4+b^4+c^4 = 9+4 = \boxed{13}$

Question.8 JEE Main 2022

Let p and q be two real numbers such that $p+q=3$ and $p^4+q^4=369$. Then the value of $\left(\frac{1}{p}+\frac{1}{q}\right)^{-2}$ is:

(A) 2 (B) 4 (C) 16 (D) 64

$p+q=3$ $\quad p^4+q^4=369$

$p^4+q^4=(p^2)^2+(q^2)^2$

$=(p^2+q^2)^2-2p^2q^2$

$369=\left((p+q)^2-2pq\right)^2-2(pq)^2$

$369=(9-2pq)^2-2(pq)^2$

$369=(9-2t)^2-2t^2$

$369=81+4t^2-36t-2t^2$

$2t^2-36t-288=0$

$t^2-18t-144=0$

$t=24 \quad t=-6$

$pq=24 \quad pq=-6$

$Req=\left(\frac{1}{p}+\frac{1}{q}\right)^{-2}$

$=\left(\frac{p+q}{pq}\right)^{-2}=\left(\frac{3}{24}\right)^{-2}=\left(\frac{1}{8}\right)^{-2}=64$

$\left(\frac{3}{-6}\right)^{-2}=\left(\frac{-1}{2}\right)^{-2}=(-2)^2=4$

Question.9

If x, y, z are real numbers such that $\boxed{x^2 + y^2 + 2z^2 - 4x + 2z - 2yz + 5 = 0}$, find the value of $x - y - z$.

(A) 0 (B) 2 (C) 4 (D) 5

$$\square_1^2 + \square_2^2 + \square_3^2 = 0$$

$$x^2 + y^2 + 2z^2 - 4x + 2z - 2yz + 5 = 0$$

$$(x^2 - 4x + 4) + (y^2 - 2yz + z^2) + (z^2 + 2z + 1) = 0$$

$4x$

$2(2)(x)$

$$(x-2)^2 + (y-z)^2 + (z+1)^2 = 0$$

$\boxed{x = 2}$ $\boxed{y = z}$ $\boxed{z = -1}$

$\therefore y = -1$

Req $= x - y - z$

$= 2 - (-1) - (-1)$

$= 4$

Point to Remember

Note 1

Property: If $x, y \in \mathbb{R}$ and $\underbrace{x^2}_{\geq 0} + \underbrace{y^2}_{\geq 0} = 0$, then it implies that $x = 0$ and $y = 0$.

Generalization: If $a_1, a_2, \ldots, a_n \in \mathbb{R}$ and $\boxed{\underbrace{a_1^2}_{\geq 0} + \underbrace{a_2^2}_{\geq 0} + \cdots + \underbrace{a_n^2}_{\geq 0} = 0,}$ then $a_1 = a_2 = \cdots = a_n = 0$.

$\therefore a_1 = a_2 = \cdots = a_n = 0$

Example: To solve $(x-1)^2 + (y-2)^2 = 0$:

- Since $(x-1)^2 \geq 0$ and $(y-2)^2 \geq 0$, their sum can only be zero if both terms are zero.

  $\therefore x = 1, \; y = 2$
- $x - 1 = 0 \implies x = 1$ and $y - 2 = 0 \implies y = 2$

Note 2 *

The square of any real number is always non-negative.

$a \in \mathbb{R}, \; a^2 \geq 0$

$$\boxed{a^2 \geq 0} \quad \text{for any } a \in \mathbb{R}$$

Question.1

If $x^2 + y^2 + 4z^2 - 6x - 2y - 4z + 11 = 0$, where $x, y, z \in \mathbb{R}$, then the value of xyz is:

(A) $\frac{3}{2}$ (B) 4 (C) 6 (D) 3

$$x^2 + y^2 + 4z^2 - 6x - 2y - 4z + 11 = 0$$

$$4z^2 - 4z$$

$$(2z)^2 - 2(2z)(1)$$

$$\left(x^2 - 2(3)x + (3)^2\right) + \left(y^2 - 2(y)(1) + (1)^2\right) + \left((2z)^2 - 2(2z)(1) + (1)^2\right)$$

$$(x-3)^2 + (y-1)^2 + (2z-1)^2 = 0$$

$$x = 3 \quad y = 1 \quad z = \frac{1}{2}$$

$$\text{Req} = xyz = 3(1)\left(\frac{1}{2}\right) = 3/2$$

Question.2

If $\underbrace{|x^2 - 1|}_{\geq 0} + \underbrace{(x-1)^2}_{\geq 0} + \underbrace{\sqrt{x^2 - 3x + 2}}_{\geq 0} = 0$, then the value of x is:

(A) 1 (B) 4 (C) -2 (D) None of these

$\therefore\ |x^2-1| = 0 \implies x^2 - 1 = 0 \implies x^2 = 1 \implies \boxed{x \pm 1}$

$(x-1)^2 = 0 \implies x = 1$

$\sqrt{x^2 - 3x + 2} = 0 \implies x^2 - 3x + 2 = 0 \implies x = 1, x = 2$

$(x-1)(x-2) = 0$

2: $-2, -1$

Ans: $\boxed{x = 1}$

Question.3 Ans:(A)

If $a, b, c \in \mathbb{R}$, then find the minimum value of the expression:

$$E = a^2 + 9b^2 + 25c^2 + 2a + 6b - 10c + 20$$

(A) 17 (B) 20 (C) 10 (D) 0

$$E = a^2 + 2a + (3b)^2 + 6b + (5c)^2 - 10c + 20$$

$$2(3b)(1) \qquad 2(5c)(1)$$

$$= (a+1)^2 - (1)^2 + (3b+1)^2 - (1)^2 + (5c-1)^2 - (1)^2 + 20$$

$$E = (a+1)^2 + (3b+1)^2 + (5c-1)^2 + 20 - 1 - 1 - 1$$

$E_{min} = 17$ when $(a+1)^2 = 0$ also $(3b+1)^2 = 0$ and $(5c-1)^2 = 0$

Perfect Square

① $a^2 + 6a + 10$

$(a+3)^2 - (3)^2 + 10$

$a^2 + 2(3)(a) + (3)^2 - (3)^2 + 10$

② $t^2 + 10t + 2$

$(t+5)^2 - (5)^2 + 2$

③ $a^2 + 7a + 9 = \left(a + \frac{7}{2}\right)^2 - \left(\frac{7}{2}\right)^2 + 9$

④ $x^2 - 20x + 1$

$x^2 - 2(10)x + (10)^2 - (10)^2 + 1$

$(x-10)^2 - (10)^2 + 1$

⑤ $x^2 - 11x + 3$

$\left(x - \frac{11}{2}\right)^2 - \left(\frac{11}{2}\right)^2 + 3$

⑥ $3x^2 - 9x + 1$

$3\left[x^2 - 3x + \frac{1}{3}\right]$

$3\left[\left(x - \frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + \frac{1}{3}\right]$

Brain Teaser Question

Find the number of integral solutions for the equation:

Main

$$\boxed{xy = 2x - y}$$

(A) 2 (B) 3 (C) 4 (D) 5

Brain Teaser Solution: Method 1

Goal: Find all integral solutions of $\boxed{xy = 2x - y}$.

1. **Rearrange the equation:**

$$\boxed{2x - xy - y = 0}$$

$2x - y(x+1)$

2. **Add and subtract 2:**

$$2x + 2 - xy - y = 2$$

$$2(x+1) - y(x+1) = 2$$

3. **Isolate the factors:**

$2 \times (-1) = -2$

$(-2) \times 1$

$(-1) \times 2$

$1 \times (-2)$

$$(x+1)(2-y) = 2$$

$$\boxed{(x+1)(y-2) = -2}$$

4. **Find integer factor pairs of -2:**

Since x and y are integers, $(x+1)$ and $(y-2)$ must be integer factors of -2. The possible pairs are:

$$(1,-2),(-1,2),(2,-1),(-2,1)$$

5. **Solve for (x, y) for each pair:**

- $x+1=1, y-2=-2 \implies (0,0)$
- $x+1=-1, y-2=2 \implies (-2,4)$
- $x+1=2, y-2=-1 \implies (1,1)$
- $x+1=-2, y-2=1 \implies (-3,3)$

✓

The four integral solutions are (0,0), (-2,4), (1,1), and (-3,3).

Brain Teaser Solution: Method 2

Goal: Find all integral solutions of $xy = 2x - y$.

1. **Isolate y terms:**

$$y + xy = 2x$$

$$xy + y = 2x$$

$$y(1 + x) = 2x$$

$$y(x+1) = 2x$$

2. **Solve for y (assuming $x \neq -1$):**

$$y = \frac{2x}{x+1}$$

$$y = \frac{2x}{(x+1)}$$

$$y = \frac{2(x+1) - 2}{x+1}$$

$$y = \frac{2x+2-2}{(x+1)}$$

$$y = \frac{2(x+1) - 2}{(x+1)} = 2 - \frac{2}{x+1}$$

$x, y \in I$

$x=1, y=1$

** $$\boxed{y = 2 - \frac{2}{x+1}}$$

3. **Find integer solutions:**
   - $x + 1 = 1 \implies x = 0 \implies y = 0$
   - $x + 1 = -1 \implies x = -2 \implies y = 4$
   - $x + 1 = 2 \implies x = 1 \implies y = 1$
   - $x + 1 = -2 \implies x = -3 \implies y = 3$

4. **Check the excluded case:** If $x = -1$

The four integral solutions are (0,0), (-2,4), (1,1), and (-3,3).