Modulus Function

mathbyiiserite

Modulus Function (Absolute Value Function)

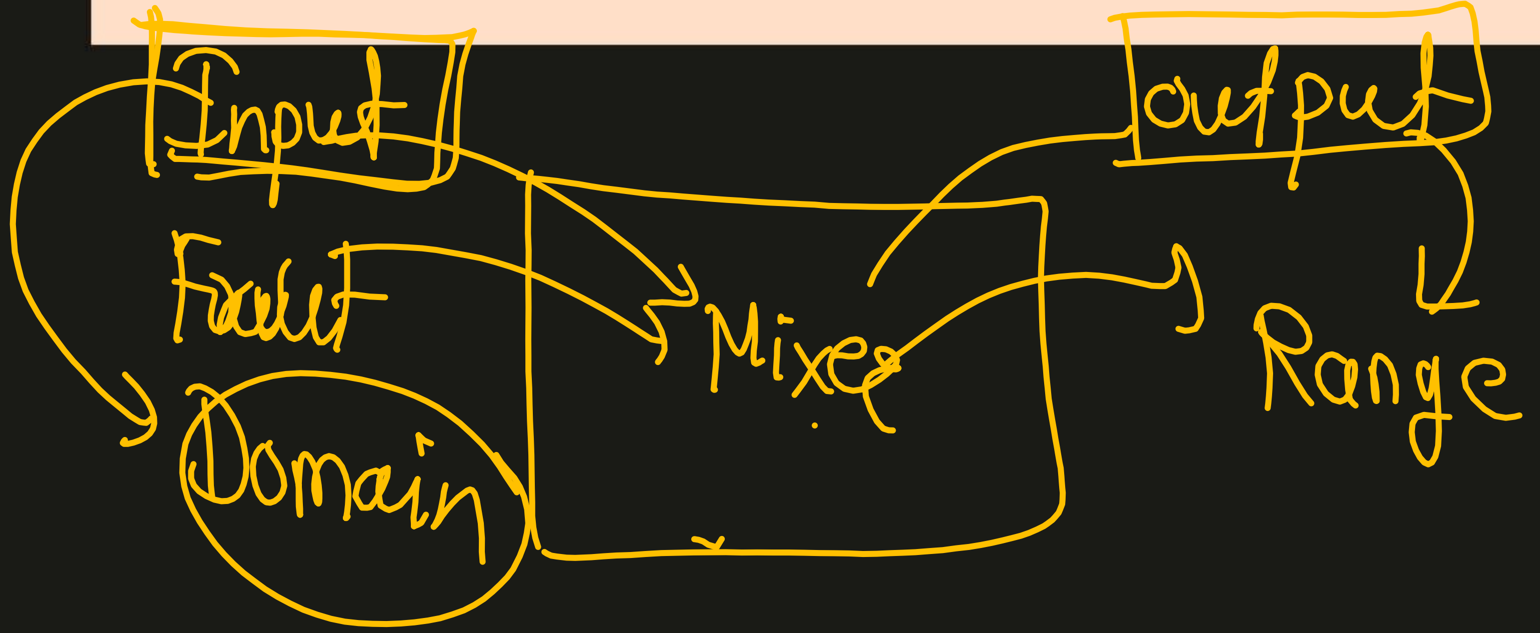
Definition

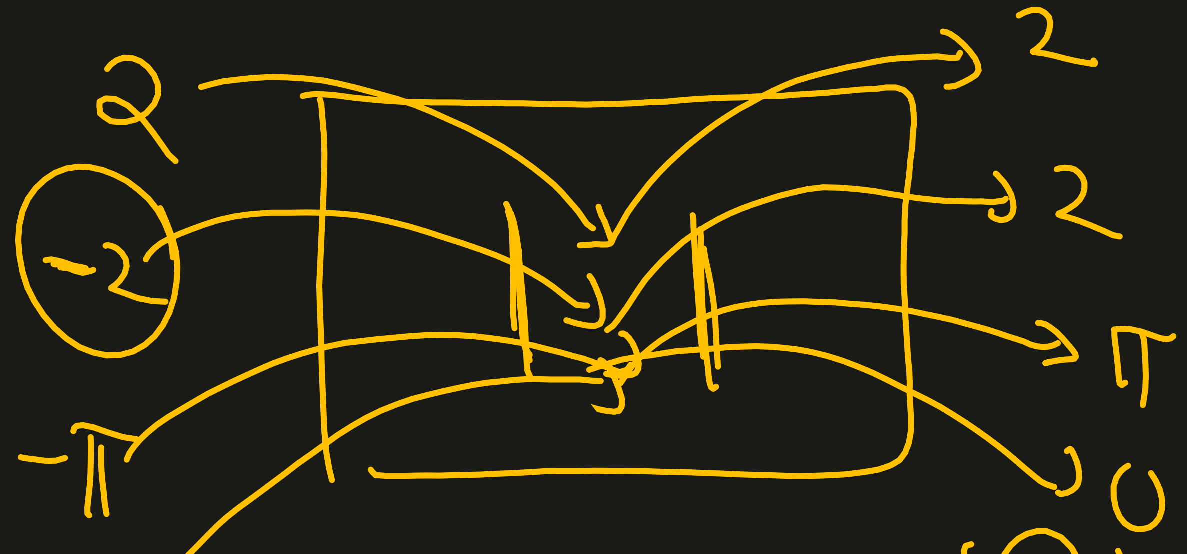
$|x| = x$ if $x \geq 0$

$|x| = -x$ if $\underline{x < 0}$

$$|x| = \begin{cases} x, & \text{if } \underline{x \geq 0} \\ -x, & \text{if } \underline{x < 0} \end{cases}$$

- **Domain:** The set of all real numbers, $\boxed{x \in \mathbb{R}.}$
- **Range:** The set of all non-negative real numbers, $\boxed{[0, \infty).}$





0

$|-5| = 5$

$|-7| = -(-7) = 7$

Modulus of any Expression

$$|\square| = \begin{cases} \square, & \text{if } \square \geq 0 \\ -\square, & \text{if } \square < 0 \end{cases}$$

$$|\blacksquare| = \begin{cases} \blacksquare \ ; & \blacksquare \geq 0 \\ -\blacksquare \ ; & \blacksquare < 0 \end{cases}$$

Practice: Defining Modulus Functions

Write the piecewise definition for each of the following modulus functions and find their critical points.

1. $|x-2|$

$$|\square| = \begin{cases} \square ; & \square \geq 0 \\ -\square ; & \square < 0 \end{cases}$$

$$|x-2| = \begin{cases} (x-2) ; & x-2 \geq 0 \\ -(x-2) ; & x-2 < 0 \end{cases} = \begin{cases} x-2 ; & x \geq 2 \\ -(x-2) ; & x < 2 \end{cases}$$

Critical pt $x=2$

2. $|2x+7|$

$$|2x+7| = \begin{cases} 2x+7 ; & 2x+7 \geq 0 \\ & x \geq -7/2 \\ -(2x+7) ; & 2x+7 < 0 \\ & x < -7/2 \end{cases}$$

Critical pt: $x = -7/2$

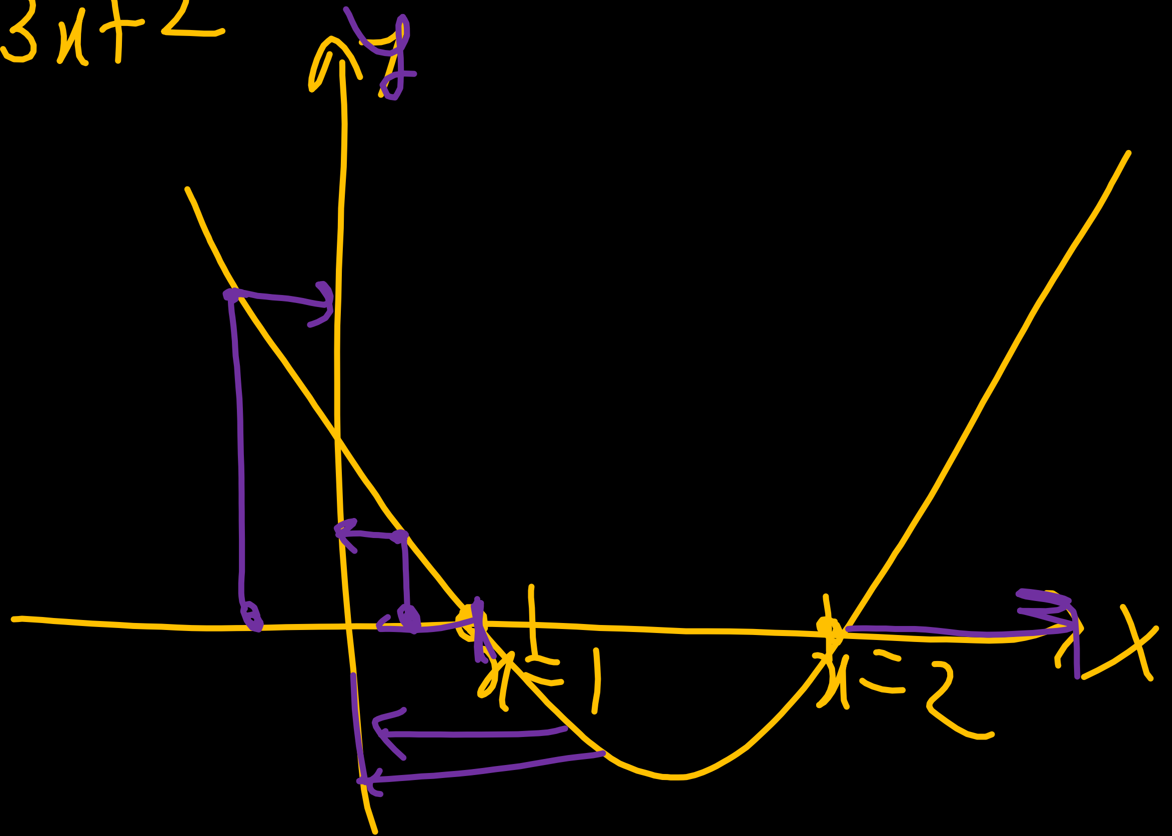
Example 3

Define the function: $|x^2 - 3x + 2|$

$$|x^2-3x+2| = \begin{cases} x^2-3x+2 ; & x^2-3x+2 \geq 0 \\ & x \in (-\infty, 1] \cup [2, \infty) \\ -(x^2-3x+2) ; & x^2-3x+2 < 0 \\ & x \in (1,2) \end{cases}$$

Critical pts $\Rightarrow x=1$ & $x=2$

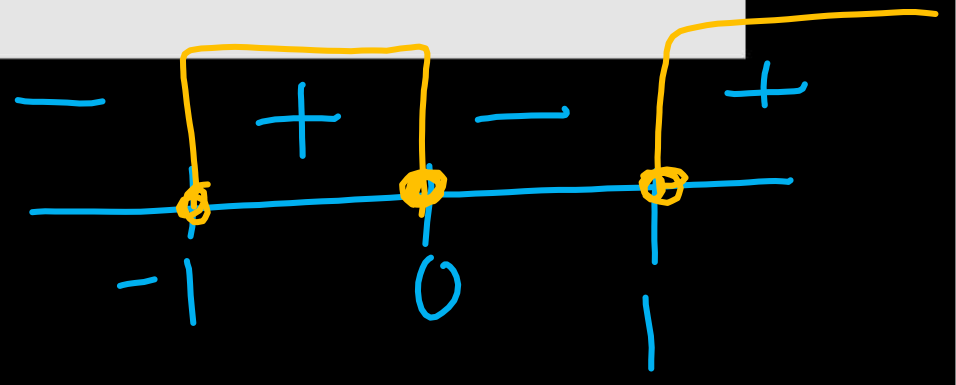
$f(x) = x^2-3x+2$



$$|\square| = \begin{cases} \square ; & \square \geq 0 \\ -\square ; & \square < 0 \end{cases}$$

Example 4

Define the function: $|x^3 - x|$



$$|x^3 - x| = \begin{cases} x^3 - x \; ; & x^3 - x \geq 0 \\ -(x^3 - x) \; ; & x^3 - x < 0 \end{cases}$$

$x(x^2-1) \geq 0$

$x(x-1)(x+1) \geq 0$

$x \in [-1, 0] \cup [1, \infty)$

$x \in (-\infty, -1) \cup (0, 1)$

Example 5

Define the function: $|2^x - 2|$

$2^x - 2 = 0$

$2^x = 2$

$x = 1$

$$|2^x - 2| = \begin{cases} 2^x - 2 & ; \quad 2^x - 2 \geq 0 \Longrightarrow x \geq 1 \\ -(2^x - 2) & ; \quad 2^x - 2 < 0 \Longrightarrow x < 1 \end{cases}$$

$-$ $+$

$x = 1$

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Example 1

Define the function: $y = |x+2| + |x-5|$

Critical pt $x = -2,\ x = 5$

Case I: $x \le -2$ | Case II: $-2 < x < 5$ | Case III: $x \geq 5$

$x = -2$ | $x = 5$

	Case I	Case II	Case III
for $\lvert x+2 \rvert$	$-$	$+$	$+$
for $\lvert x-5 \rvert$	$-$	$-$	$+$

Case I:

$y = -(x+2) - (x-5)$

$y = -2x + 3$

Case II:

$y = (x+2) - (x-5)$

$y = 7$

Case III:

$y = (x+2) + (x-5)$

$y = 2x - 3$

Example 2

Define the function: $y = |x+7| - |x-2|$

Example 3

Define the function: $y = |x - 2| + |x^2 - 5x + 6|$

Critical pt $x=2$, $x=2$, $x=3$

$6 \to -3, -2$

Case I: $x \le 2$ | Case II: $2 < x < 3$ | Case III $x \geq 3$

$x=2$ $\quad$ $x=3$

for $|x-2|$: $-$ | $+$ | $+$

for $|x^2-5x+6|$: $+$ | $-$ | $+$

$|(x-2)(x-3)|$

$(x-2)(x-3) \geq 0$

$y = -(x-2) + (x^2-5x+6)$

$y = (x-2) - (x^2-5x+6)$

$y = -x^2+6x-8$

$y = x-2+x^2-5x+6$

$y = x^2-4x+4$

Example 4

Define the function: $y = |x-1| + |x-2| + |x-3|$

Critical 1, 2, 3

Case I: $x \le 1$

$x = 1$, $x = 2$, $x = 3$

		$x=1$		$x=2$		$x=3$	
$\lvert x-1\rvert$	$-$		$+$		$+$		$+$
$\lvert x-2\rvert$	$-$		$-$		$+$		$+$
$\lvert x-3\rvert$	$-$		$-$		$-$		$+$

$y = -(x-1)-(x-2)-(x-3)$
$= -3x+6$

$y = (\cancel{x}-1)-(\cancel{x}-2)-(x-3)$
$y = -x+4$

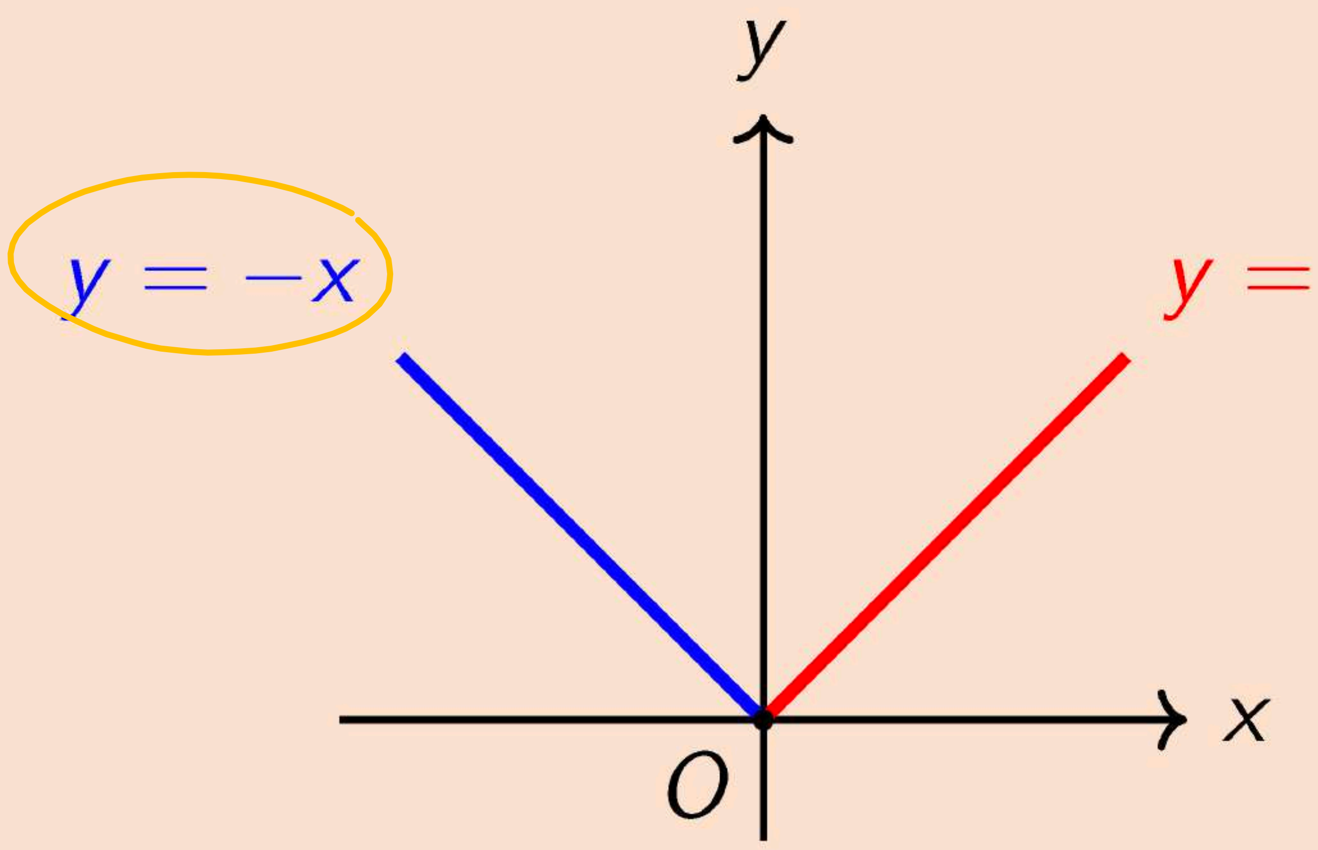
$y = x-1+\cancel{x}-2-\cancel{x}+3$
$y = x$

$y = x-1+(x-2)+(x-3)$
$y = 3x-6$

Graphs of Modulus Function

Graph of $y = |x|$

$$y = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$



Basic of Straight Line

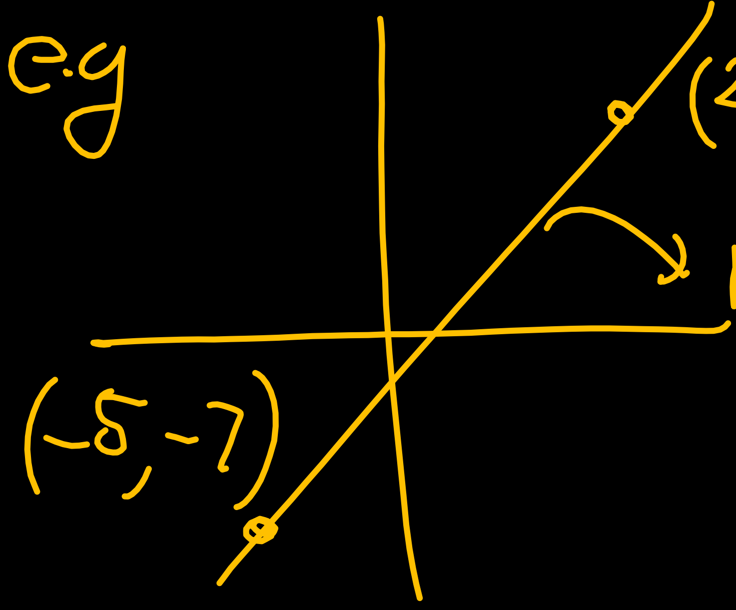
$$y = mx + c$$

$m \rightarrow$ slope

$c \rightarrow$ ~~x~~ y-intercept

$$\boxed{\text{Slope} = m = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1} = \frac{dy}{dx}}$$

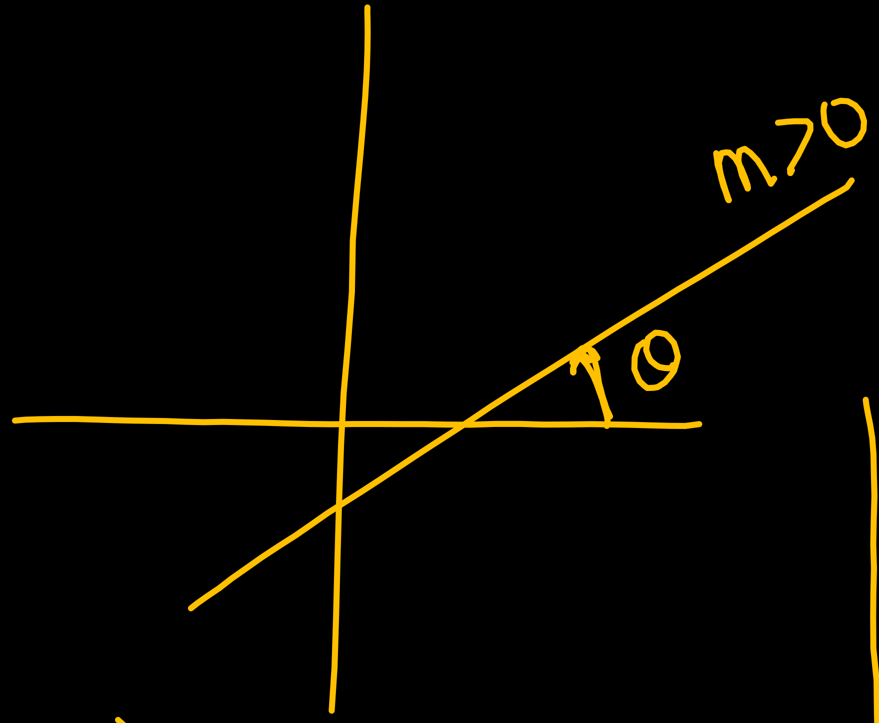
e.g

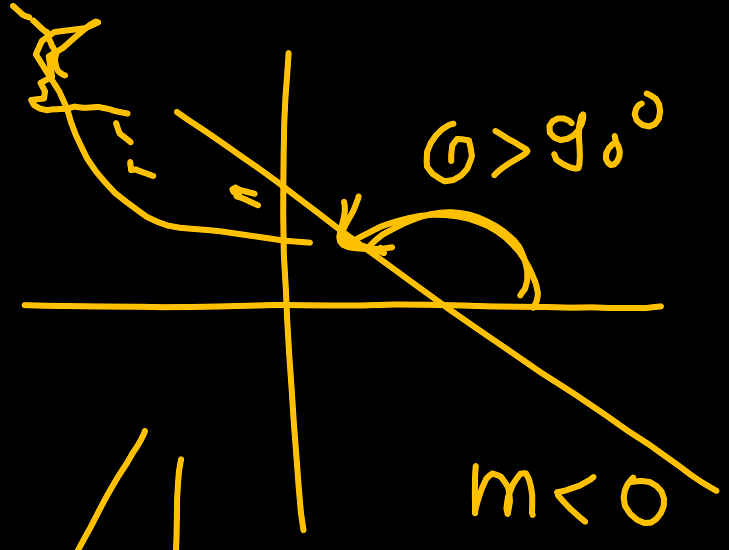


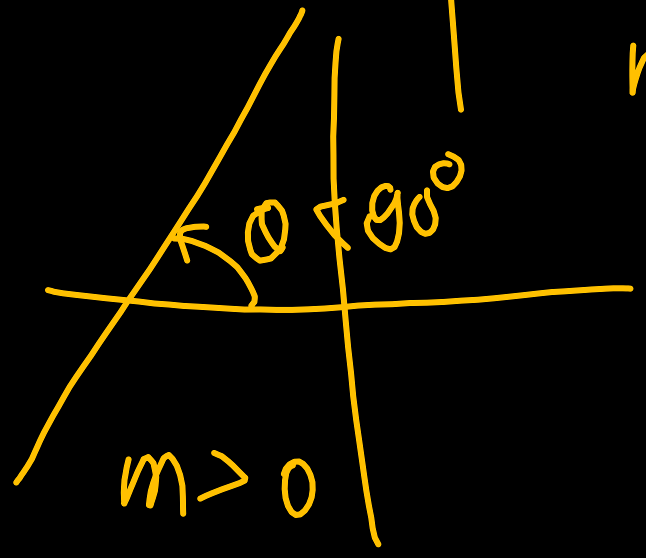
$$m = \frac{3 - (-7)}{2 - (-5)} = \frac{10}{7}$$

Slope $\rightarrow m = \tan\theta \qquad 0 \le \theta < 180^\circ$

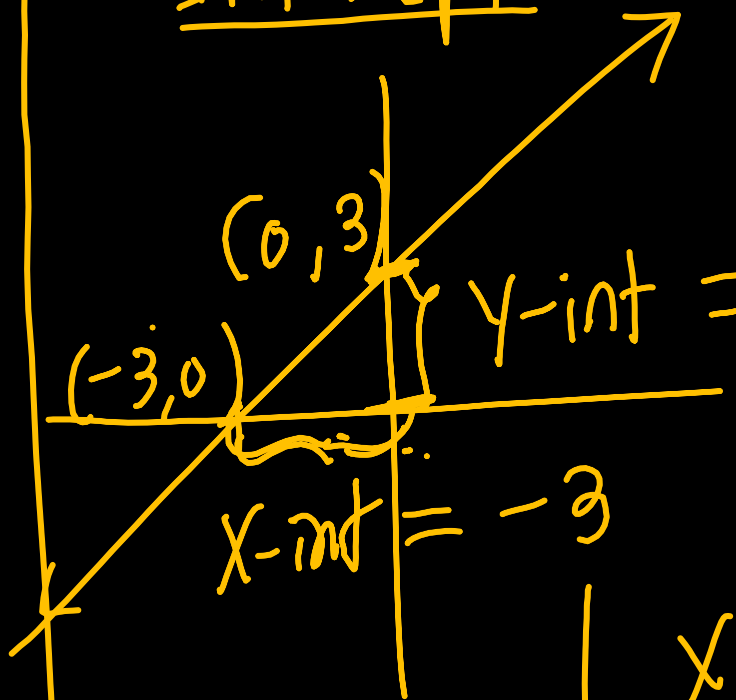
$\tan\theta < 0$	$\tan\theta > 0$
$-ve$	$+ve$
$\theta > 90^\circ$	$0 \le \theta < 90^\circ$

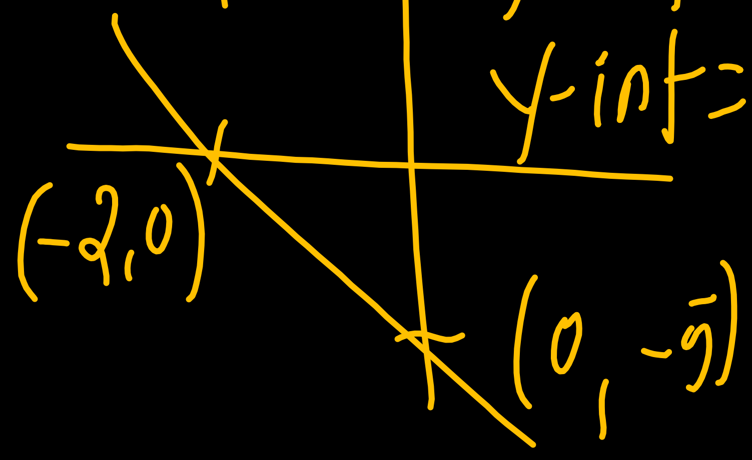






Intercept





x-int = -2

y-int = -5

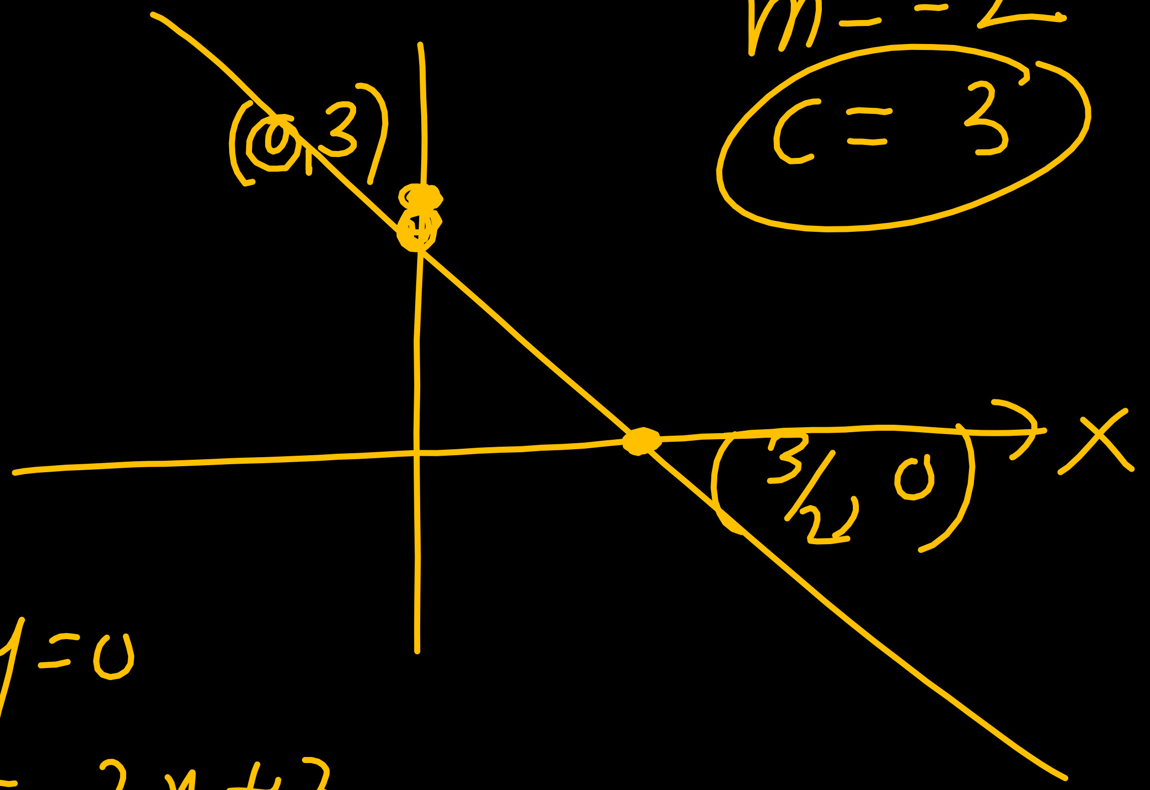
e.g., (1)

$y = mx + c$

$y = -2x + 3$

$m = -2$

$c = 3$



put $y = 0$

$0 = -2x + 3$

$2x = 3$

$x = 3/2$

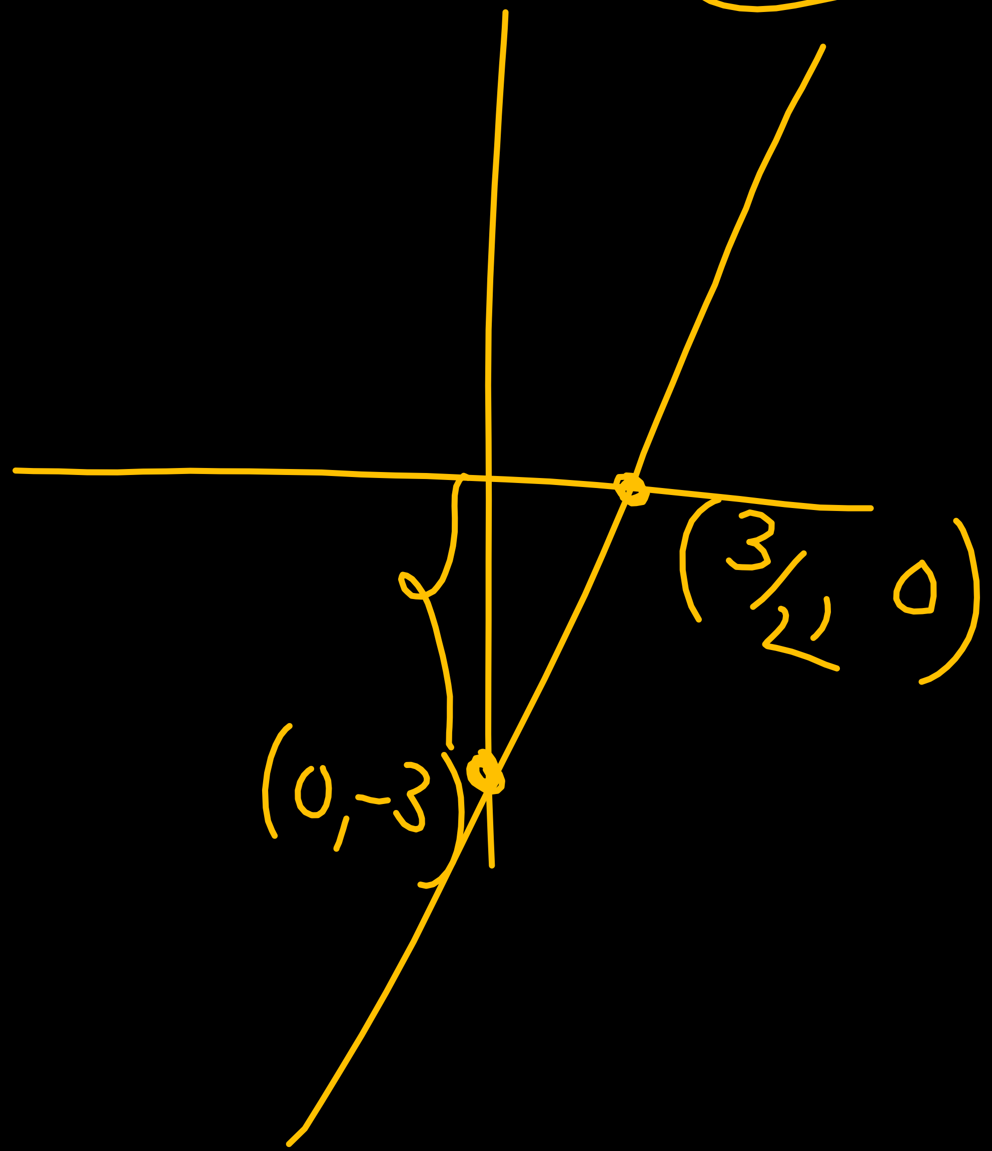
(2)

$4x - 2y = 6$

$2y = 4x - 6$

$y = 2x - 3$

$m = 2$ y-int $= -3$



Example 1

Draw the graph of the function:

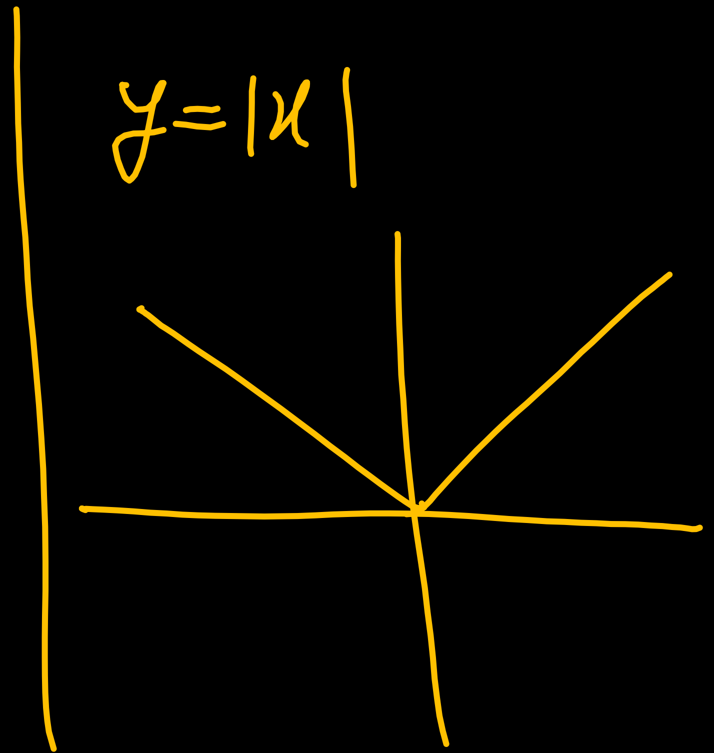
$$y = |x - 1|$$

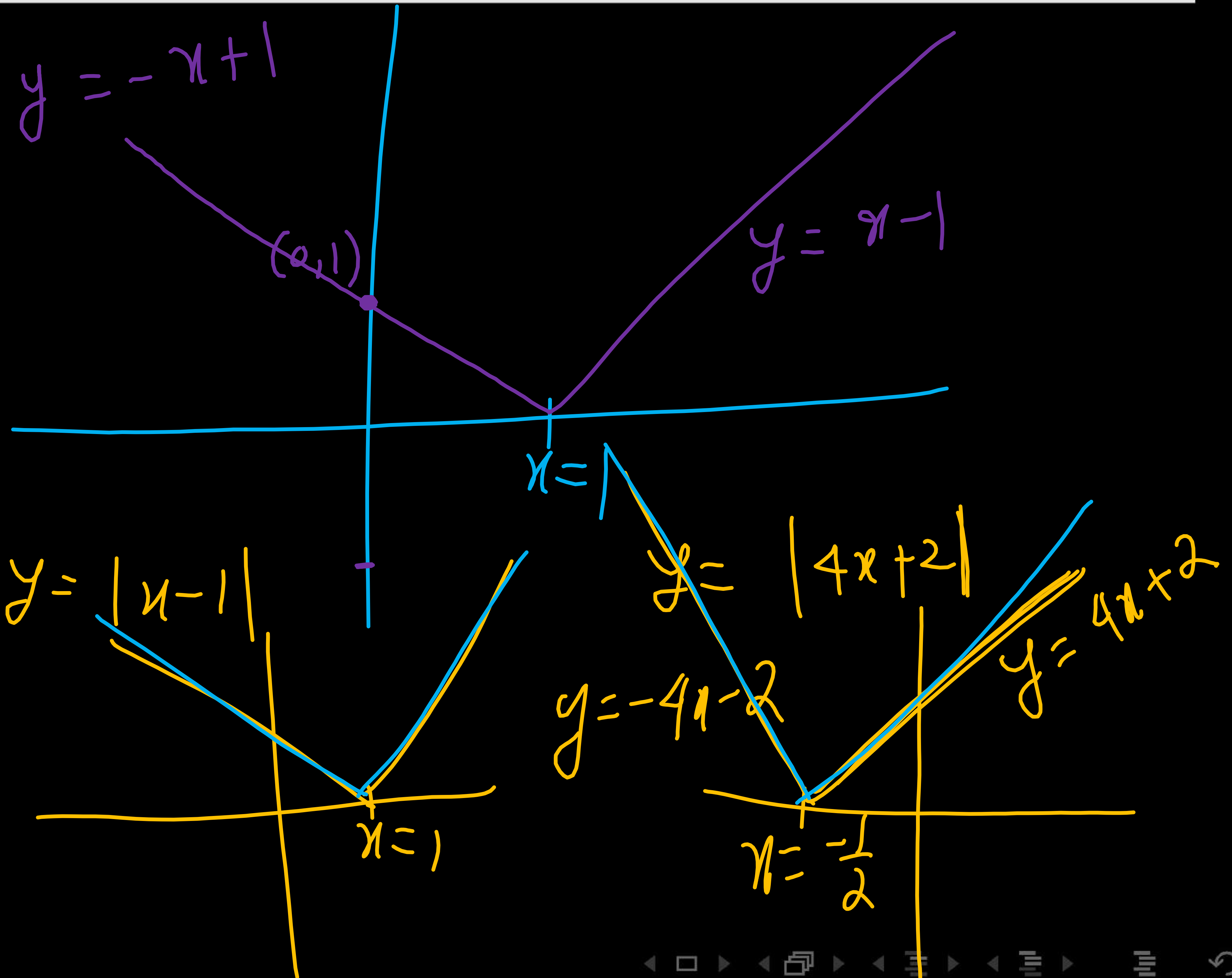
$y = x - 1$

$$y = |x-1| = \begin{cases} x-1 ; & x-1 \geq 0 \\ & x \geq 1 \\ -(x-1) ; & x-1 < 0 \\ & x < 1 \end{cases}$$

$y = -x + 1$

$m = -1$

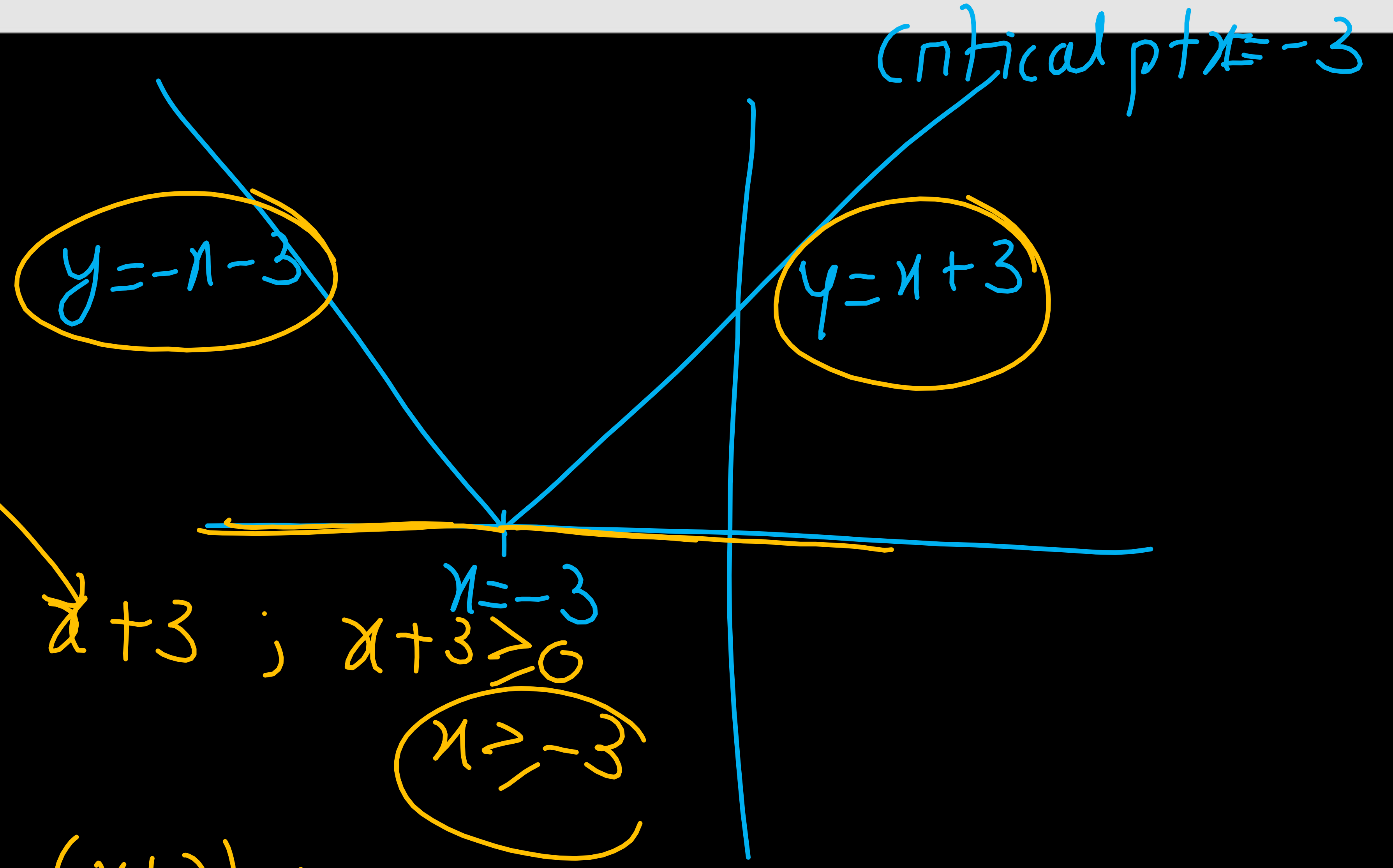




Example 2

Draw the graph of the function:

$$y = |x+3|$$



$$y = |x+3| = \begin{cases} x+3 \; ; & x+3 \geq 0 \\ & x \geq -3 \\ -(x+3) \; ; & x+3 < 0 \\ & x < -3 \end{cases}$$

Example 3

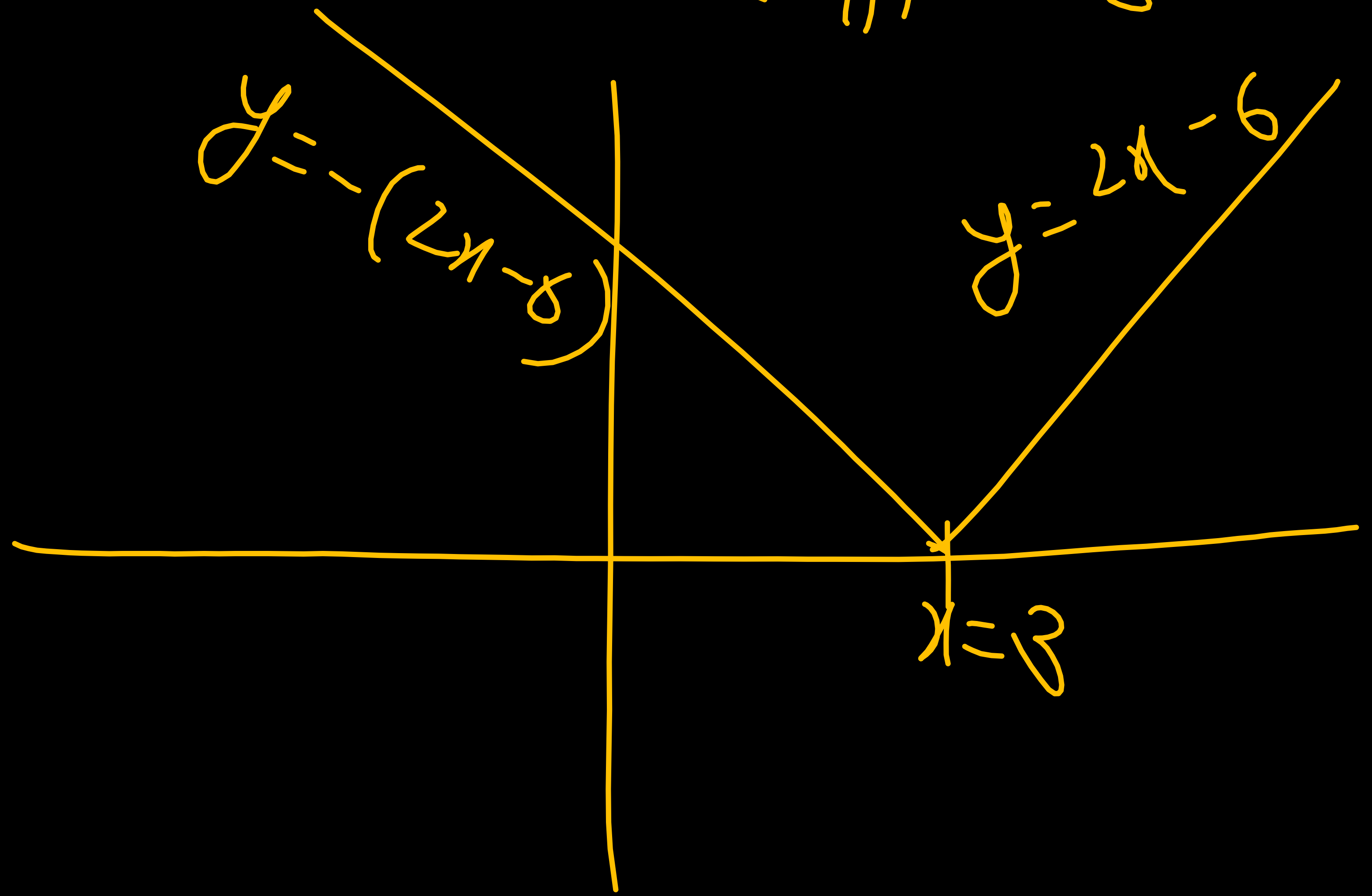
Draw the graph of the function:

$$y = |2x - 6|$$

$$2x - 6 = 0$$

$$x = \frac{6}{2}$$

Critical pt $x = 3$



Example 4

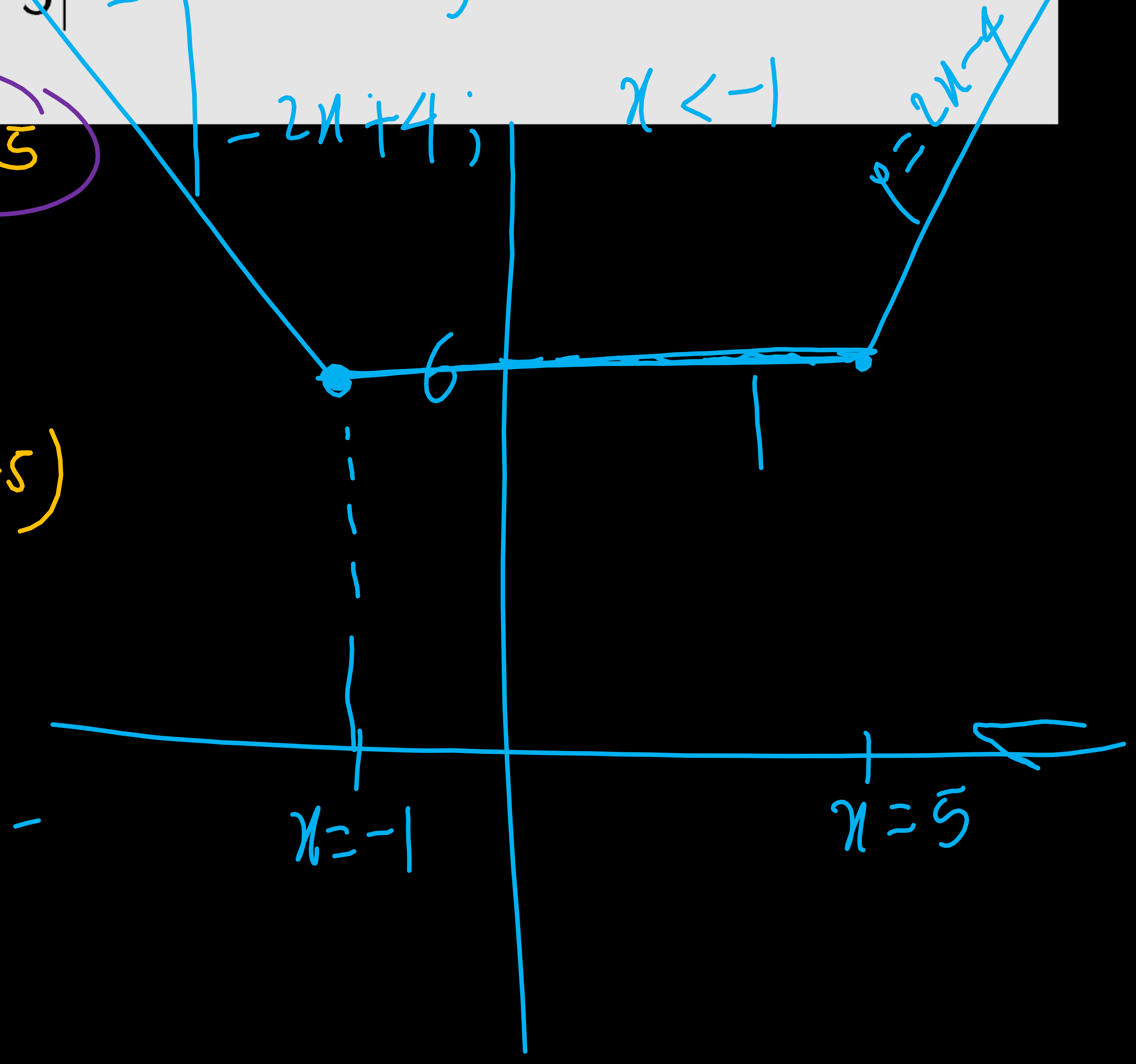
Draw the graph of the function:

$$y = |x+1| + |x-5| = \begin{cases} 2x-4 ; & x \geq 5 \\ 6 ; & -1 \le x < 5 \\ -2x+4 ; & x < -1 \end{cases}$$

$x = -1$ $\quad$ $x = 5$

$x=-1$ $\quad$ $x=5$

$y = -(x+1)-(x-5)$	$y = (\cancel{x}+1)-(\cancel{x}-5)$	$y = x+1+(x-5)$
$y = -2x+4$	$y = 6$	$y = 2x-4$



$y = |x+1| + |x-5|$

Shortcut

$f(x) = |x+1| + |x-5|$

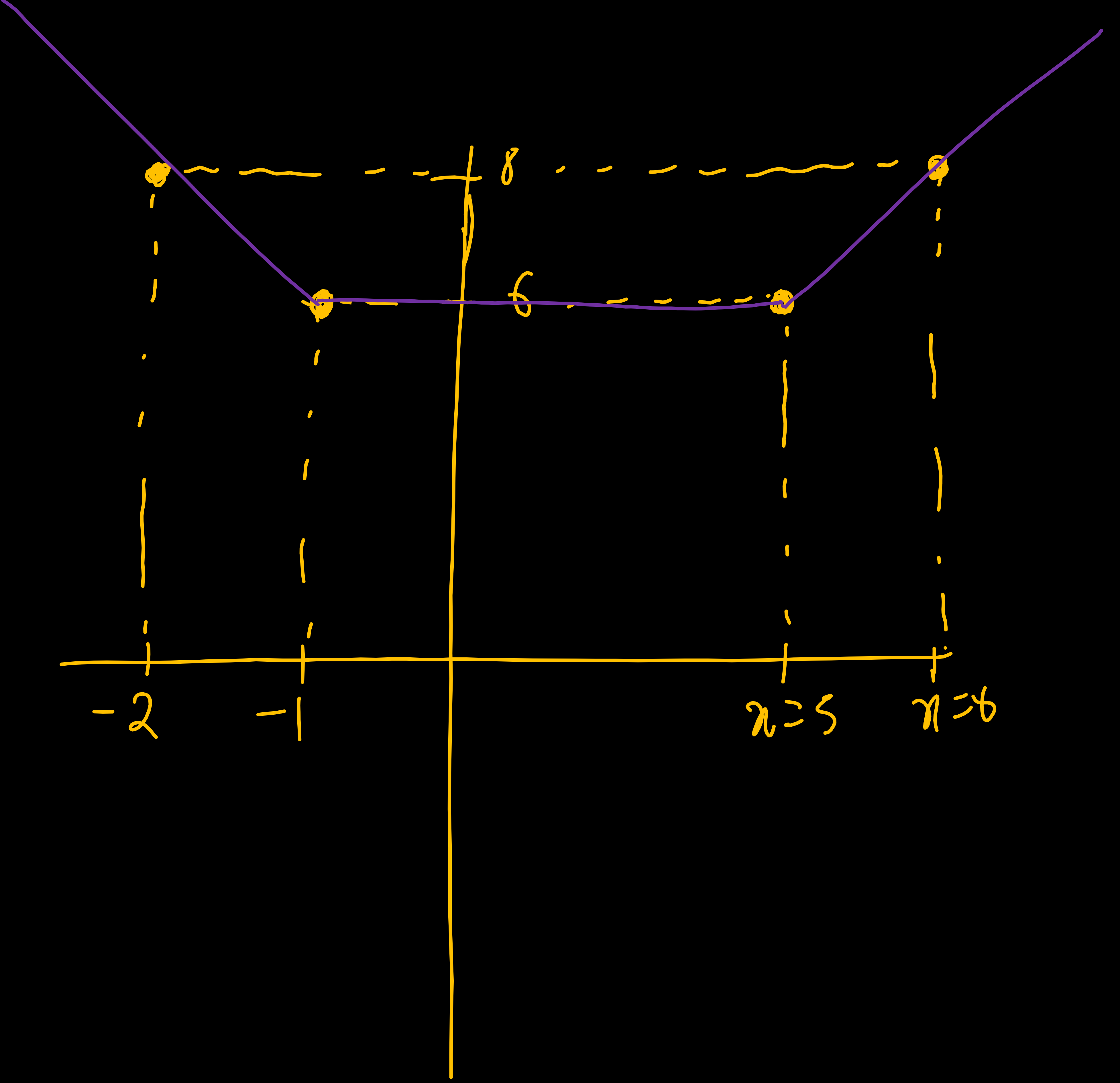
Find value of f(x) at critical pts.

$f(-2) = 1+7 = 8$

$f(-1) = 6$

$f(5) = 6$

$f(6) = 7+1 = 8$



Example 5

Draw the graph of the function:

$$f(x)\ y = |x-1| + |x-2| + |x-3|$$

Critical pt $x=1,\ x=2,\ x=3$

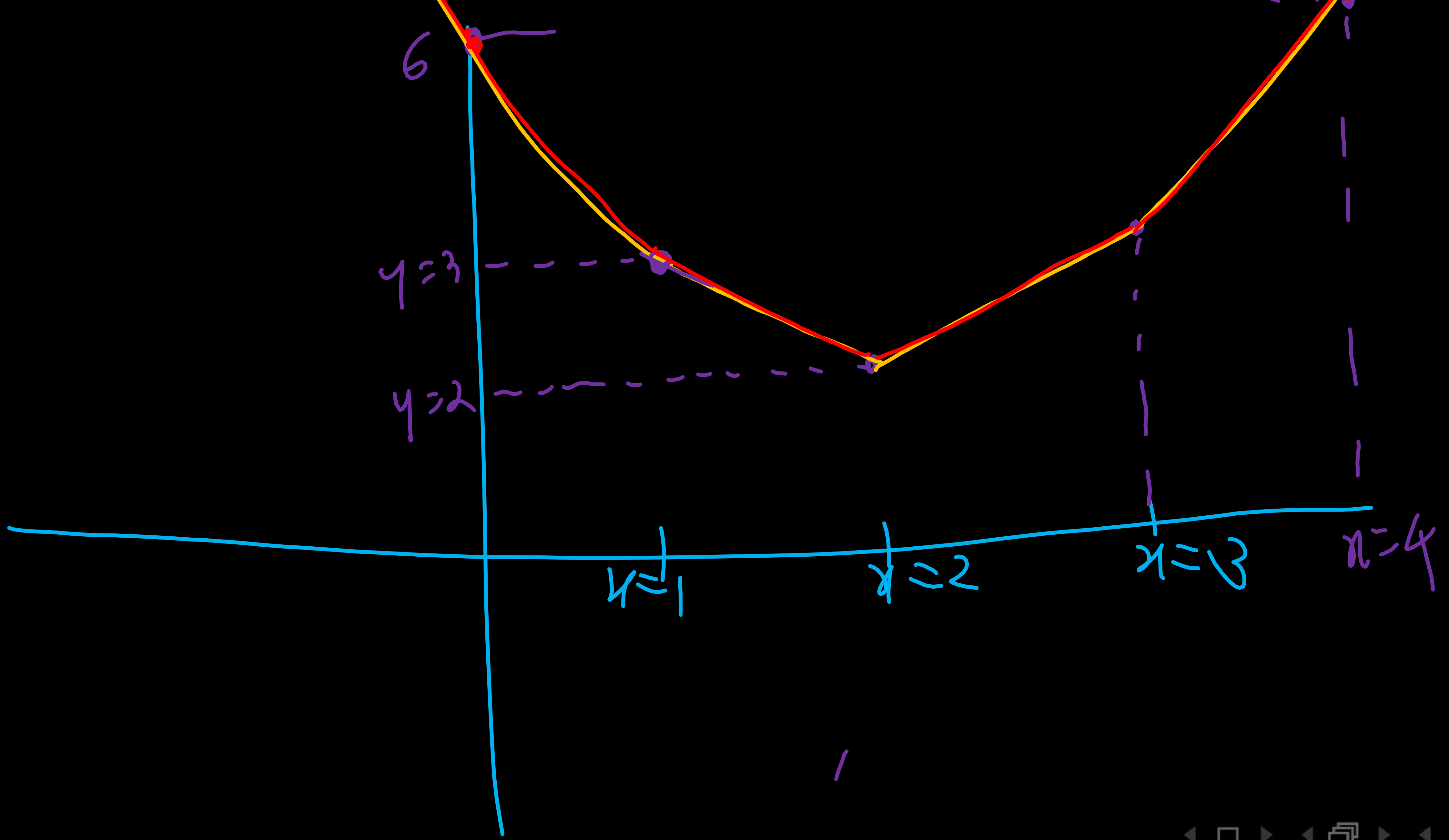
$f(0) = 6$

$f(1) = 3$

$f(2) = 2$

$f(3) = 3$

$f(4) = 6$



Example 6

Draw the graph of the function:

$$y = |x - 2| - |x - 4|$$

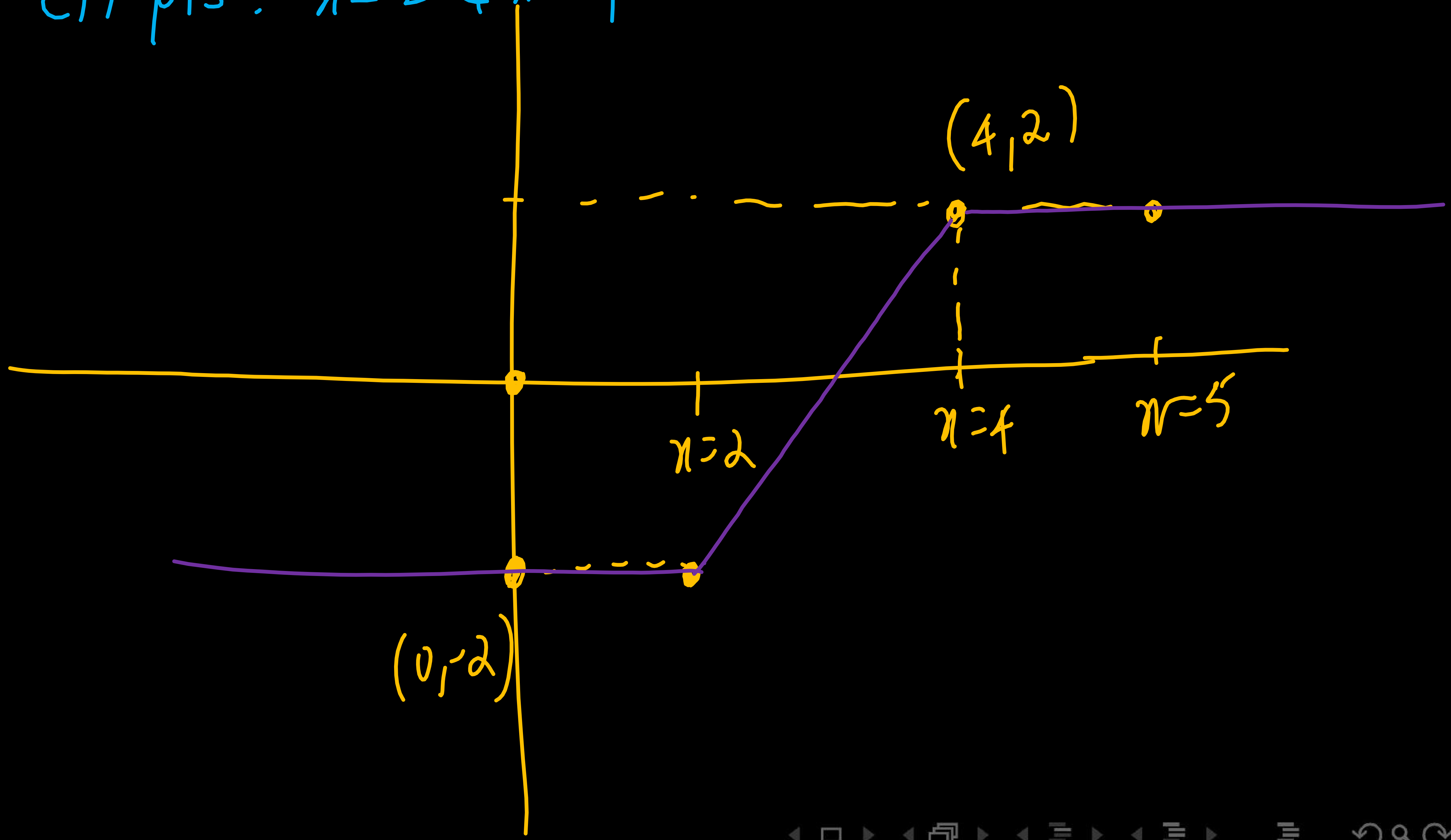
crt pts : $x = 2$ & $x = 4$

$f(0) = 2 - 4 = -2$

$f(2) = -2$

$f(4) = 2$

$f(5) = 3 - 1 = 2$



Example 7

Draw the graph of the function:

$7 - 11 + 1$

$$y = |x+1| - |2x-1| + |x-5|$$

$f(-1) = -|-3| + |-1-5| = -3 + 6 = 3$

Critical pts: $x = -1$, $x = \frac{1}{2}$, $x = 5$

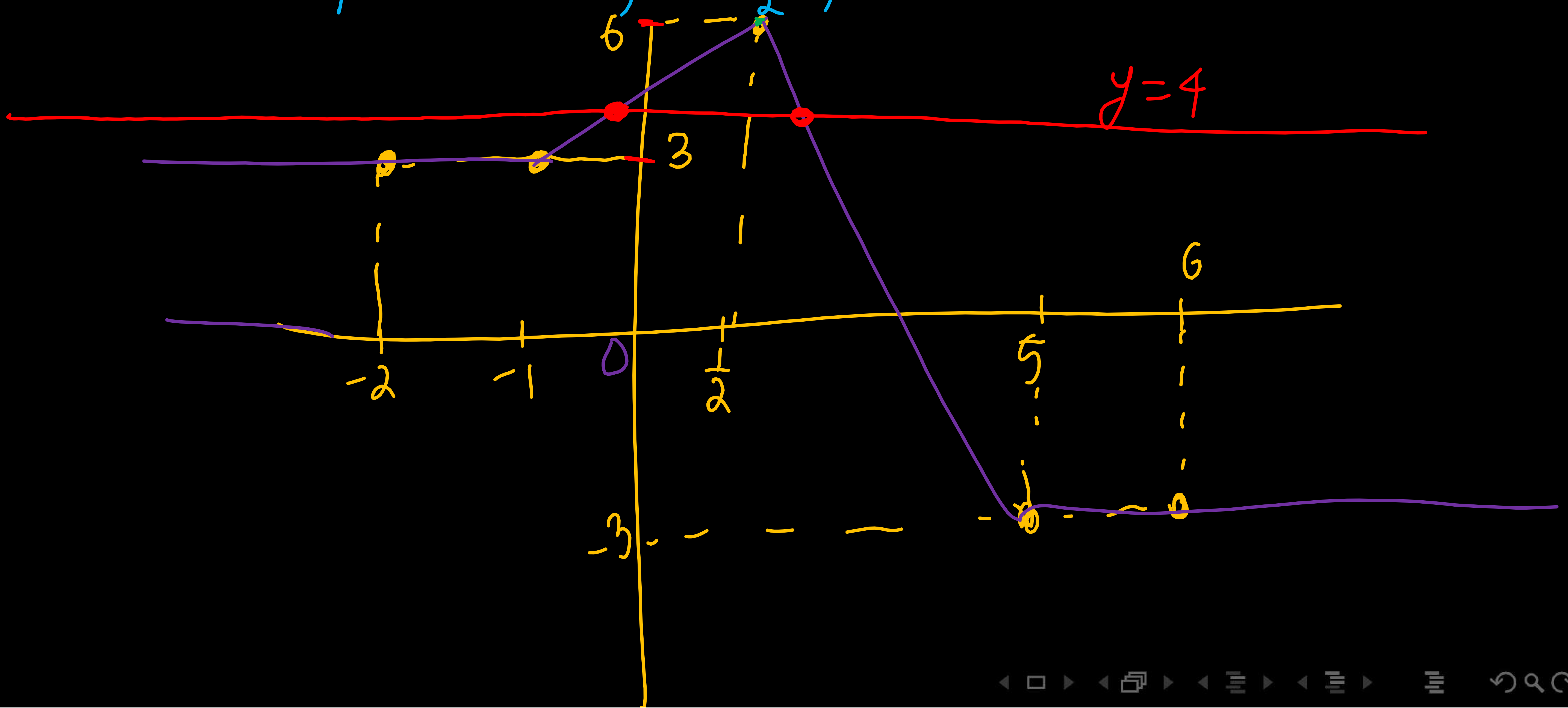
$f(-2) = 3$

$f(-1) = 3$

$f(\frac{1}{2}) = 6$

$f(5) = -3$

$f(6) = -3$



Example 8

Find the number of solutions for the equation:

$$\underbrace{|x+1| - |2x-1| + |x-5|}_{f(x)} = \underbrace{\lambda}_{g(x)} = 4$$

For $\lambda = 4$

Also find value of λ for which given equation have exactly one solution? $\rightarrow \lambda \in (-3,3) \cup \{6\}$

$y = f(x) = |x+1| - |2x-1| + |x-5|$

$y = g(x) = 4$

$\therefore$ Number of sol^n for $\lambda = 4 \longrightarrow n = 2$

for $\lambda = -4 \longrightarrow n = 0$

$\lambda = 3 \longrightarrow$ Infinite sol^n

$\lambda = 2 \longrightarrow n = 1$

Concept: Finding the Number of Solutions

Graphical Method for Solving Equations

To find the number of solutions for an equation of the form:

$$\boxed{f(x) = g(x)}$$

Graph $y = f(x)$

Graph $y = g(x)$

We can use a graphical approach.

Draw the graph of $y = f(x)$ and the graph of $y = g(x)$ on the same coordinate plane.

The total number of points where the two graphs intersect will be the number of solutions to the equation.

Example

Ans:3

Find the number of solutions for the equation:

3

$$|x+1| - |x| + |2x-1| = 2^x$$

$2x = 2^x$

$y = 2^x$

$2^{1/2} = \sqrt{2} = 1.41$

$f(x) = |x+1| - |x| + |2x-1|$

critical pts $-1, 0, \frac{1}{2}$

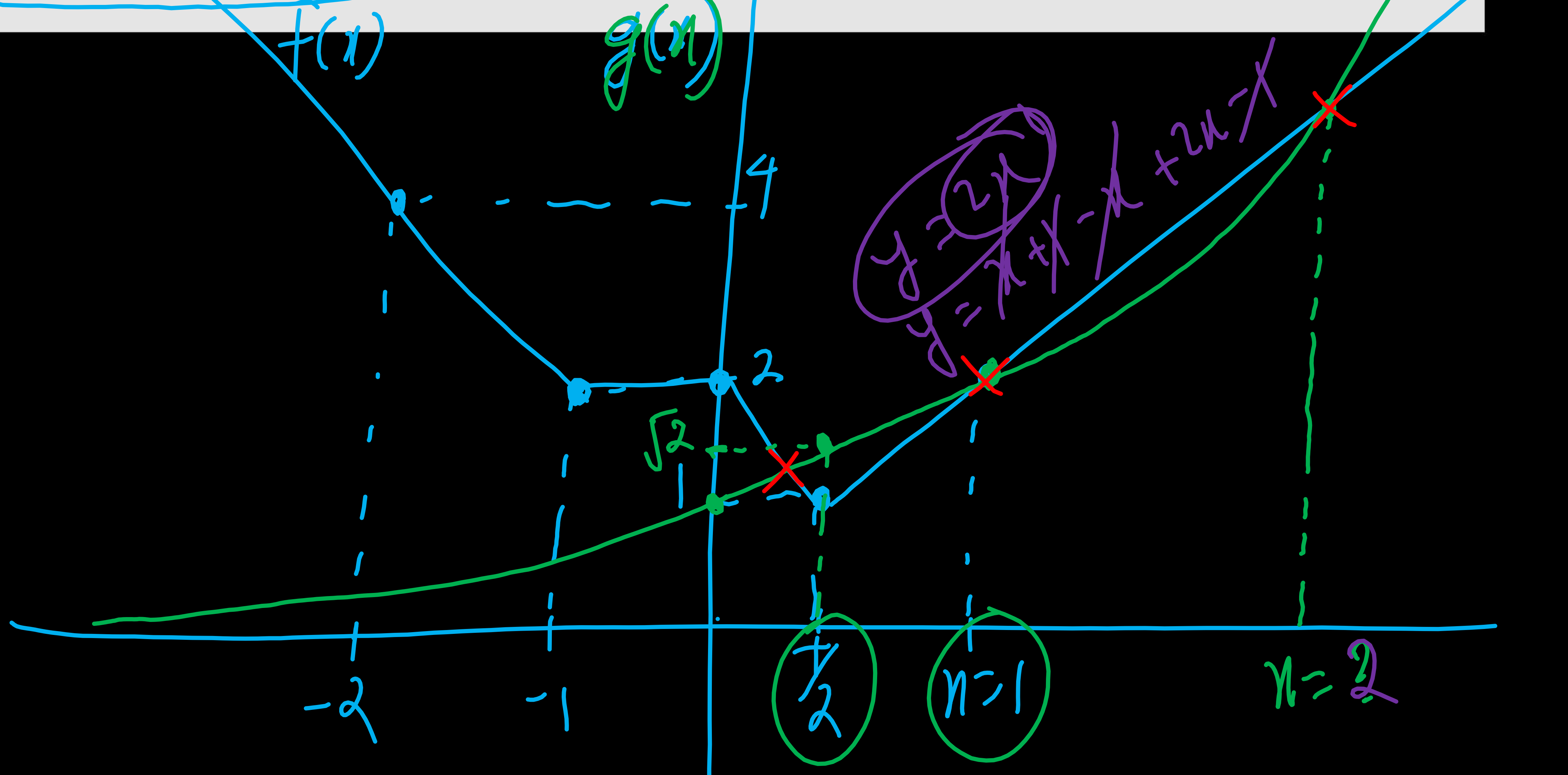
$f(-2) = 1 - 2 + 5 = 4$

$f(-1) = -1 + 8 = 2$

$f(0) = 2$

$f(\tfrac{1}{2}) = 3/2 - 1/2 = 1$

$f(1) = 2$



Ans: No. of sol^n = 3

Properties of Modulus Function

Basic Properties

1. $|x| \geq 0$ (Modulus is always non-negative)
2. $|x| = 0 \iff x = 0$

   $|\square| = 0 \iff \square = 0$
3. $|x| = a$, for $a > 0 \implies x = \pm a$

   $|x| = 2 \implies x = \pm 2$

   $|\square| = a \implies \square = a$ or $\square = -a$
4. $|x| = a$, for $a < 0 \implies x \in \emptyset$ (No solution)

   $|x| = -2 \implies x \in \emptyset$

   eg $|x| = 4 \implies x = \pm 4$

   $|4| = 4 \quad |-4| = 4$

   $|x| = -2 \implies x \in \emptyset$
5. $\sqrt{x^2} = |x|$
6. $x^2 = |x|^2$

$x^2 - 5|x| + 6 = 0$

$(|x|)^2 - 5|x| + 6 = 0$

$\sqrt{(1+\sin x)^2} = |1+\sin x|$

Properties of Modulus Function

$|x-2| = 3$

$x=-1$ $x=5$

$x=-1$ $x=0$ $x=2$ $x=5$

Properties Involving Operations

7. $|xy| = |x||y|$

8. $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$

9. $|x| = |-x|$ $\longrightarrow$ $|5x-6-x^2| = |-(\boxed{x^2-5x+6})| = |x^2-5x+6|$

$|-\square| = |\square|$

10. $|x-a| = |a-x|$

*11. $|x+y| \leq |x| + |y|$ (Triangle Inequality)

$|x+y| \neq |x|+|y|$

Geometrical meaning $|x| = 2$

Comple No

$|x-0| = 2$

$x=-2$ $x=0$ $x=2$

Example 1

Solve the equation:

$$|\boxed{}| = 4$$

$$|x - 3| = 4 \longrightarrow x - 3 = \pm 4$$

$$x - 3 = 4 \qquad\qquad x - 3 = -4$$

$$x = 7 \qquad\qquad x = -1$$

Example 2

Solve the equation:

$$|x - 1| = -2$$

Negative

$\therefore x \in \phi$

Example 3

Solve the equation:

$$|x^2 - 10| = 6$$

$x^2 - 10 = 6$

$x^2 = 16$

$x = \pm 4$

$x^2 - 10 = -6$

$x^2 = 4$

$x = \pm 2$

Example 4

Solve the equation:

$$| \boxed{} | = 2$$

$$\underline{||x+2|-3|} = 2$$

$$|x+2|-3 = 2 \qquad\qquad |x+2|-3 = -2$$

$$|x+2| = 5 \qquad\qquad |x+2| = 1$$

$$x+2 = 5 \qquad x+2 = -5 \qquad x+2 = 1 \qquad x+2 = -1$$

$$x = 3 \qquad x = -7 \qquad x = -1 \qquad x = -3$$

Example 5

Solve the equation:

$$||3x - 4| + 3| = 7$$

$|3x-4| + 3 = 7$

$|3x-4| = 4$

$3x-4=4$

$3x=8$

$x = 8/3$

$3x-4=-4$

$3x=0$

$x=0$

$|3x-4| + 3 = -7$

$|3x-4| = -10$

No sol^n

Example 6

Solve the equation:

$$|||x-3|-3|-3| = 3$$

$$||x-3|-3|-3 = 3 \qquad\qquad ||x-3|-3|-3 = -3$$

$$||x-3|-3| = 6 \qquad\qquad \underline{||x-3|-3|} = 0$$

$$|\square| = 0 \Rightarrow \square = 0$$

$$|x-3|-3 = 6 \qquad |x-3|-3 = -6 \qquad |x-3|-3 = 0$$

$$|x-3| = 9 \qquad |x-3| = -3 \qquad |x-3| = 3$$

No sol^n

$$x-3 = 9 \qquad x-3 = -9 \qquad x-3 = 3 \qquad x-3 = -3$$

$$x = 12 \qquad x = -6 \qquad x = 6 \qquad x = 0$$

Example 7

Solve the equation:

$$\frac{|x-4|}{|x+3|} = 5$$

$$\left|\frac{x-4}{x+3}\right| = 5 \qquad |\cancel{B}| = 5$$

$$\frac{x-4}{x+3} = 5 \qquad\qquad \frac{x-4}{x+3} = -5$$

$$x-4 = 5x+15 \qquad\qquad x-4 = -5x-15$$

$$-19 = 4x \qquad\qquad 6x = -11$$

$$x = -\frac{19}{4} \qquad\qquad x = -11/6$$

Example 8

Solve the equation: $x^2 = |x|^2$

$$(x-4)^2 - 3|x-4| - 4 = 0$$

$\boxed{\ }^2 = |\boxed{\ }|^2$

$$(|x-4|)^2 - 3|x-4| - 4 = 0$$

let $|x-4| = t$

$$t^2 - 3t - 4 = 0$$

-4

$-4 \quad +1$

$t = 4 \qquad t = -1$

$|x-4| = 4 \qquad |x-4| = -1$

$x - 4 = 4 \qquad x - 4 = -4 \qquad$ No soln

$x = 8 \qquad x = 0$

Example 9

Solve the equation:

$$x^2 - 3|x| - 10 = 0$$

let $|x| = t$

$\therefore \; x^2 = |x|^2 = t^2$

$t^2 - 3t - 10 = 0$

$|x| = 5 \qquad |x| = -2$

$x = \pm 5 \qquad$ No sol^n

Example 10

Solve the equation:

$$|x^2 + 3x - 4| = |x^2 - 1|$$

Type $|a| = |b| \Rightarrow a = \pm b$

$a = b$ $\quad$ $a = -b$

Case I $\quad a = b$

$\cancel{x^2} + 3x - 4 = \cancel{x^2} - 1$

$3x = 3$

$x = 1$

Case II : $a = -b$

$x^2 + 3x - 4 = -(x^2 - 1)$

$2x^2 + 3x - 5 = 0$ $\quad -10$

$+5 \quad -2$

$2x^2 - 2x + 5x - 5 = 0$

$2x(x-1) + 5(x-1) = 0$

$x = 1 \quad 2x + 5 = 0$

$x = -5/2$

Example 11

Solve the equation:

$$|x^2 - 5x + 6| = 5x - x^2 - 6$$

Type

$$|\square| = -\square$$

$$\Rightarrow \quad \square \le 0$$

$$|\square| = \begin{cases} \square & ; \ \square > 0 \\ -\square & ; \ \square \le 0 \end{cases}$$

$$|x^2 - 5x + 6| = -(x^2 - 5x + 6)$$

$$\therefore \ x^2 - 5x + 6 \le 0$$

$$(x-2)(x-3) \le 0$$

$$x \in [2, 3]$$

Example 12

Solve the equation:

$$|x-5|+|x-4|=1$$

Critical pt $x=4,\ x=5$

Case I $x \le 4$ | Case II $4 < x < 5$ | Case III $x \ge 5$

$x=4$ 4.1   4.2 $x=5$

Case I:

$-(x-5)-(x-4)=1$

$-2x=-8$

$x=4$

Case II:

$\cancel{-}(x-5)+\cancel{(}x-4)=1$

$1=1$

True

$\therefore\ x \in (4,5)$ is my Ans

Case III:

$(x-5)+(x-4)=1$

$2x-9=1$

$2x=10$

$x=5$

Example 13

Solve the equation:

$$|2x + 1| + 3x = 11$$

Ans: $x = 2$

*

critical pt $x = -\frac{1}{2}$

Case I : $x \le -\frac{1}{2}$

Case II : $x > -\frac{1}{2}$

$-$

$x = -\frac{1}{2}$

$+$

$-(2x+1) + 3x = 11$

$x = 12$

Reject

$2x + 1 + 3x = 11$

$5x = 10$

$x = 2$ ✓

**

Both the cases ⟹ Union

Within the case ⟹ Intersection

$x \in \phi$

Example 14

If $x^2 - |x - 3| - 3 = 0$, then find the possible values of $|x|$.

$|-3| = 3$

$|2| = 2$

$\therefore$ Ans: $\{2, 3\}$

Critical pt $x = 3$

Case I: $x \le 3$

$x^2 + (x-3) - 3 = 0$

$x^2 + x - 6 = 0$

$-6 \to +3, -2$

$x = -3$ ✓ $\quad x = 2$ ✓

Case II: $x > 3$

$x^2 - (x-3) - 3 = 0$

$x^2 - x = 0$

$x(x-1) = 0$

$x = 0$ ✗ $\quad x = 1$ ✗ Reject

Modulus Inequality

$|\square| \le a \Rightarrow -a \le \square \le a$ $\quad$ $|\square| \ge a \Rightarrow \square \in (-\infty,-a] \cup [a,\infty)$

Type-I: $|x| \le a$

If a is a positive constant:

- $|x| < a \iff -a < x < a$
- $|x| \le a \iff -a \le x \le a$

If a is a negative constant:

- $|x| < a \implies$ No solution $(x \in \emptyset)$

$|x| < -2 \Rightarrow x \in \phi$

If a is zero:

- $|x| < 0 \implies x \in \emptyset$ ✓
- $|x| \le 0 \implies x = 0$

$|x| < 0$ or $|x| = 0 \Rightarrow x = 0$

Type-II: $|x| \ge a$

If a is a positive constant:

- $|x| > a \iff x > a$ or $x < -a$
- $|x| \ge a \iff x \ge a$ or $x \le -a$

If a is a negative constant:

- $|x| > a \implies$ All real numbers $(x \in \mathbb{R})$

$|x| > -3 \Rightarrow x \in R$

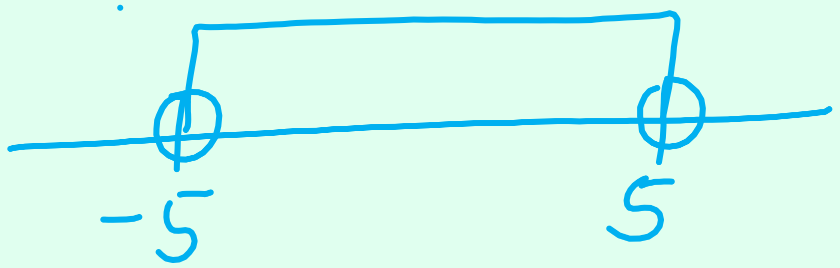
If a is zero:

- $|x| > 0 \implies x \in \mathbb{R} - \{0\}$
- $|x| \ge 0 \implies x \in \mathbb{R}$

$0 > 0$ False

Type-I Examples ($|x| \leq a$)

1. $|x| < 5$ $\Longrightarrow$ $x \in (-5, 5)$



$-5 < x < 5$

2. $|x| \leq 7$

$-7 \leq x \leq 7$

3. $|x| \leq -5$

$x \in \phi$

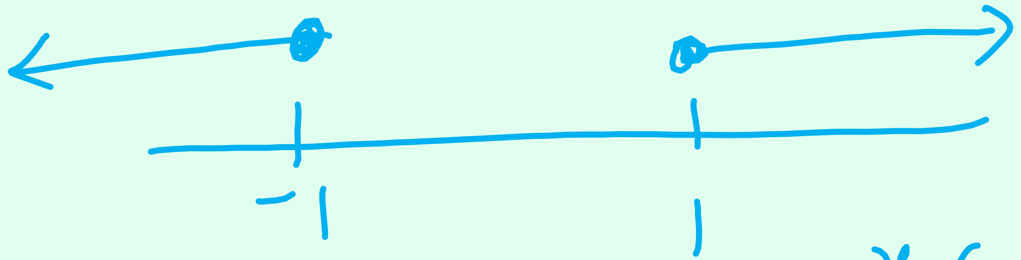
$|-3.1| = 3.1 > 3$

Type-II Examples ($|x| \geq a$)

1. $|x| > 3$

$x \in (-\infty, -3) \cup (3, \infty)$

2. $|x| \geq 1$



$x \in (-\infty, -1] \cup [1, \infty)$

3. $|x| \geq -2$

$x \in R$

Type-I Examples ($|x| \leq a$)

4. $\left|\frac{x-2}{x+1}\right| \leq 0$

$|\boxed{\ }| \leq 0 \Rightarrow |\boxed{\ }| = 0 \Rightarrow \boxed{\ } = 0$

$\frac{x-2}{x+1} = 0 \quad \therefore x = 2$

5. $|x| < 0$

$x \in \phi$

Type-II Examples ($|x| \geq a$)

4. $\left|\frac{x+3}{x-1}\right| \geq 0$

$|\boxed{\ }| \geq 0 \Rightarrow \boxed{\ } \in R$

$\frac{x+3}{x-1} \in R \Rightarrow x \in R - \{1\}$

$\therefore x \neq 1$

5. $|x| > 0$

$x \in R - \{0\}$

$|x-2| > 0$

$x \in R - \{2\}$

Type-I Examples ($|x| \le a$)

6. $|x-3| < 5$

$|\square| < a \Rightarrow -a < \square < a$

$-5 < x-3 < 5$

$-5+3 < x-\cancel{3}+\cancel{3} < 5+3$

$\boxed{-2 < x < 8}$

7. $|2x+1| \le 7$

$-7 \le 2x+1 \le 7$

$-8 \le 2x \le 6$

$\boxed{-4 \le x \le 3}$

Type-II Examples ($|x| \ge a$)

6. $|x+2| > 3$

$|\square| > a \Rightarrow \square \in (-\infty, -a) \cup (a, \infty)$

$x+2 > 3$ OR $x+2 < -3$

Union.

$x > 1$ $\qquad$ $x < -5$

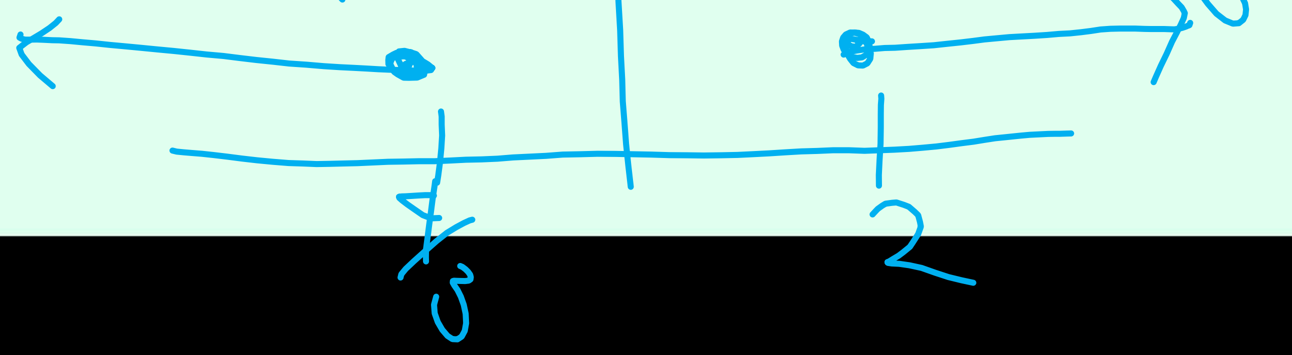
$x \in (-\infty, -5) \cup (1, \infty)$

7. $|3x-5| \ge 1$

$3x-5 \ge 1$ OR $3x-5 \le -1$

$3x \ge 6$ $\qquad$ $3x \le 4$

$x \ge 2$ $\qquad$ $x \le 4/3$



Type-I Examples ($|x| \le a$)

8. $|x^2 - 4| \le 5$ Ans: $x \in [-3, 3]$

*** $\underbrace{-5 \le x^2}-4 \le 5$

$-5 \le x^2 - 4$ and $x^2 - 4 \le 5$

Intersection

$x^2 \ge -1$ $\quad$ $x^2 \le 9$

$x \in R$ $\quad$ $x \in [-3, 3]$

9. $|x + 4| < -2$

$|\boxtimes| < -2 \Rightarrow x \in \phi$

Type-II Examples ($|x| \ge a$)

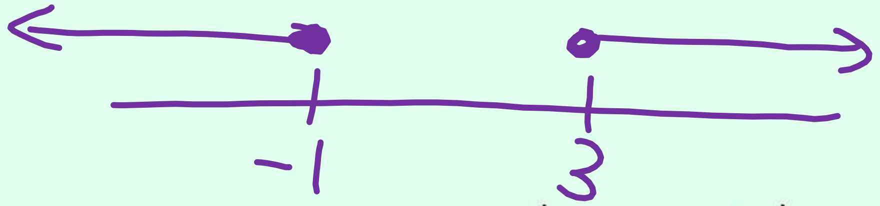
8. $|x^2 - 2x - 1| \ge 2$ Ans: $x \in (-\infty, -1] \cup [3, \infty) \cup \{1\}$

$x^2 - 2x - 1 \ge 2$ OR $x^2 - 2x - 1 \le -2$

Union

$x^2 - 2x - 3 \ge 0$ $\quad$ $x^2 - 2x + 1 \le 0$

$(x-3)(x+1) \ge 0$ $\quad$ $(x-1)^2 \le 0$ $\quad$ $\boxtimes^2 \le 0$



$\therefore x = 1$ $\quad$ $\Rightarrow \boxtimes = 0$

9. $|x - 1| > -3$

$\underbrace{|\boxtimes|}_{\ge 0} > -3$

$x \in R$

Example 1

Solve the inequality:

$$||x-2|-3| \le 3$$

$$|\square| \le 3$$

$$-3 \le \square \le 3$$

$$-3 \le |x-2|-3 \le 3$$

$$0 \le |x-2| \le 6$$

$0 \le |x-2|$ and $|x-2| \le 6$

$x \in \mathbb{R}$ — (I)

$-6 \le x-2 \le 6$

$-4 \le x \le 8$ — (II)

$\therefore$ (I) $\cap$ (II) $x \in [-4, 8]$

Example 2

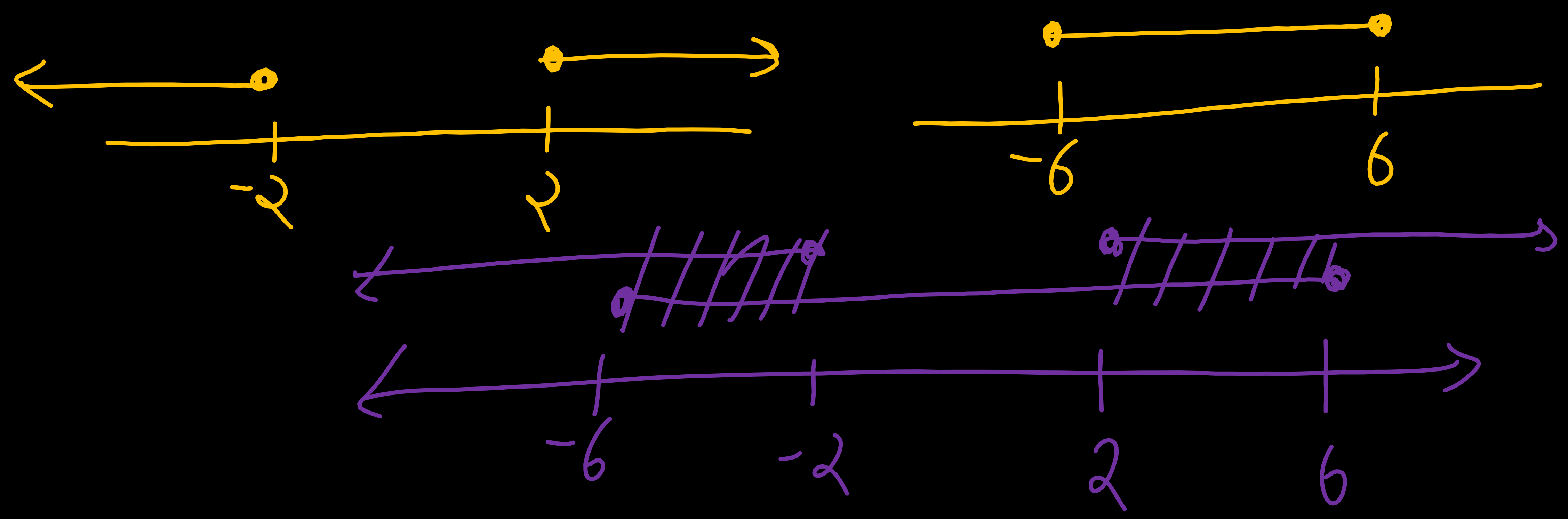
Solve the inequality:

$$2 \le |x| \le 6$$

$2 \le |x|$ and $|x| \le 6$

Intersection

$|x| \geq 2$



$x \in [-6,-2] \cup [2,6]$

Example 3

Solve the inequality:

$$||x-3|-3|<3$$

Example 4

Solve the inequality:

$$\frac{|x+3|+x}{x+2} > 1$$

Case I $\cup$ Case II (I) $\cup$ (II)

$x \in (-5,-2) \cup (-1,\infty)$

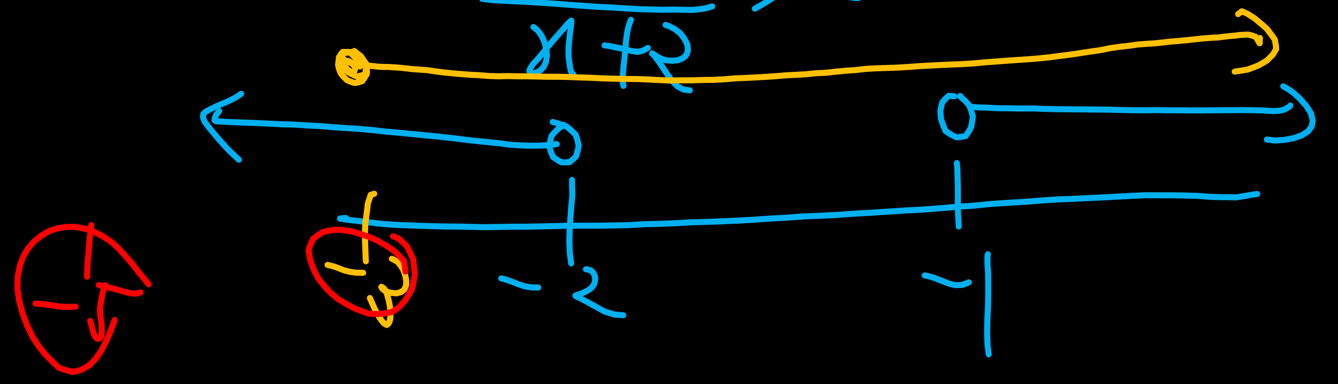
crit $x=-3$

Case I : $x \geq -3$

$$\frac{x+3+x}{x+2} > 1$$

$$\frac{2x+3}{x+2} - 1 > 0$$

$$\frac{x+1}{x+2} > 0$$



Case I Ans:

$x \in [-3,-2) \cup (-1,\infty)$ — (I)

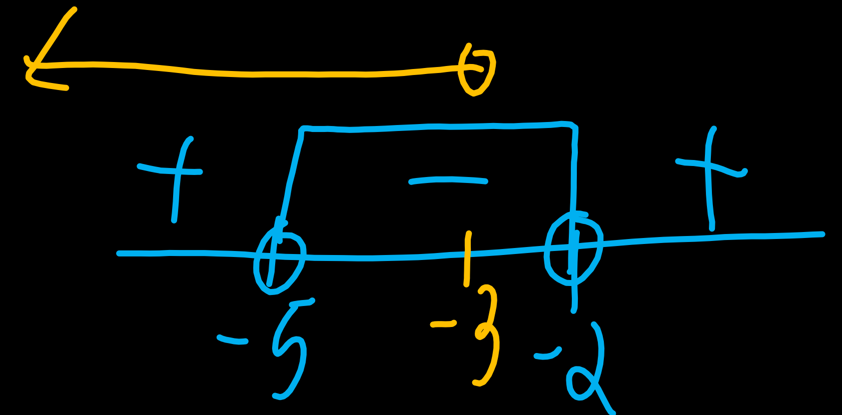
Case II : $x < -3$

$$\frac{-(x+3)+x}{x+2} > 1$$

$$\frac{-3}{x+2} - 1 > 0$$

$$\frac{-x-5}{x+2} > 0$$

$$\frac{x+5}{x+2} < 0$$



Case II Ans:

$x \in (-5,-3)$ — (II)

Example 5 (JEE 2002)

The set of all real numbers x for which

$$x^2 - |x+2| + x > 0$$

is:

critical pt $x=-2$

$\text{Ans} = (1) \cup (2)$

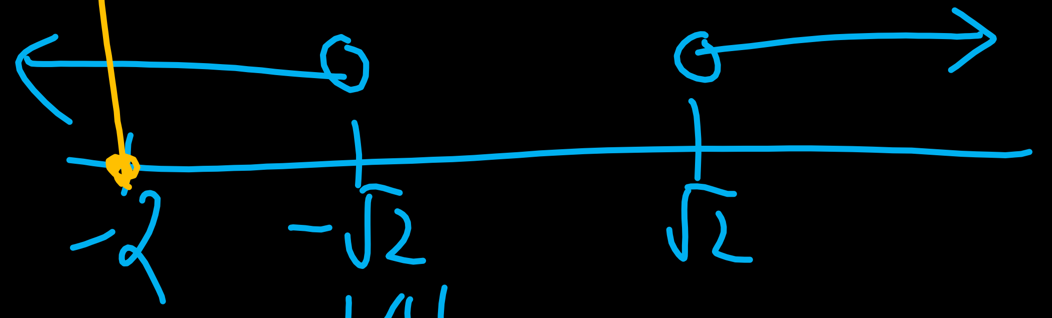
$x \in (-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

Case I: $x \geq -2$

$x^2 - (x+2) + x > 0$

$x^2 - 2 > 0$

$(x-\sqrt{2})(x+\sqrt{2}) > 0$



$x \in [-2, -\sqrt{2}) \cup (\sqrt{2}, \infty)$ — (1)

Case II: $x \leq -2$

$x^2 + x + 2 + x > 0$

$x^2 + 2x + 2 > 0$

$D < 0 \quad D = (2)^2 - 4(2) = -4$

$\therefore a > 0 \ \& \ D < 0 \Rightarrow ax^2 + bx + c > 0$

$x \in R$

Case II Ans: $x \leq -2$ — (2)

Example 6

Solve the inequality:

$$|x-1|+|x-2| \geq 4$$

Critical pts : $x=1$, $x=2$

Case I : $x \leq 1$

$-(x-1)-(x-2) \geq 4$

$-2x \geq 1$

$x \leq -\frac{1}{2}$

Case II : $\boxed{1 < x < 2}$

$(\cancel{x}-1)-(\cancel{x}-2) \geq 4$

$-1+2 \geq 4$

$1 \geq 4$

False

$\therefore$ No soln

Case III : $x \geq 2$

$(x-1)+(x-2) \geq 4$

$2x-3 \geq 4$

$2x \geq 7$

$x \geq \frac{7}{2}$

Example 7

Solve the inequality:

$$|x-1| \leq |x^2 - 2x + 1|$$

$|a| \geq |b|$

Square on both side

$|a|^2 \geq |b|^2$

$a^2 \geq b^2$

$a^2 - b^2 \geq 0$

$(a-b)(a+b) \geq 0$

Square .

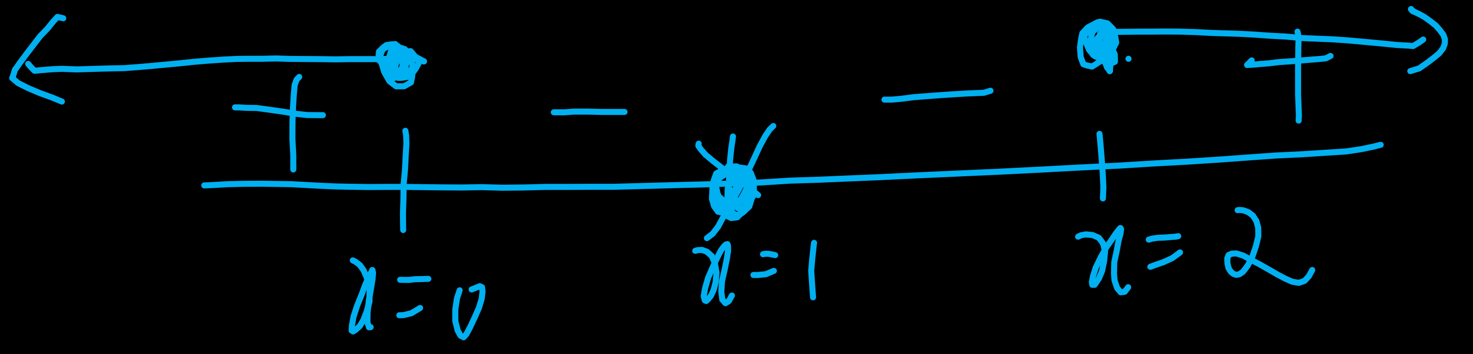
$(x-1)^2 \leq (x^2 - 2x + 1)^2$

$(x-1)^2 \leq ((x-1)^2)^2$

$(x-1)^4 - (x-1)^2 \geq 0$

$(x-1)^2 \left[(x-1)^2 - 1 \right] \geq 0$

$(x-1)^2 x (x-2) \geq 0$



$x \in (-\infty, 0] \cup [2, \infty) \cup \{1\}$

Example 8 (JEE Main 2022) Ans:(B)

Let $A = \{x \in R : |x+1| < 2\}$ and $B = \{x \in R : |x-1| \geq 2\}$. Then which one of the following statements is NOT true?

(A) $A - B = (-1, 1]$

(B) $B - A = R - (-3, 1)$

(C) $A \cap B = (-3, -1]$

(D) $A \cup B = R - [1, 3)$

$|\square| < a$

$-2 < x+1 < 2$

$A: \; -3 < x < 1$

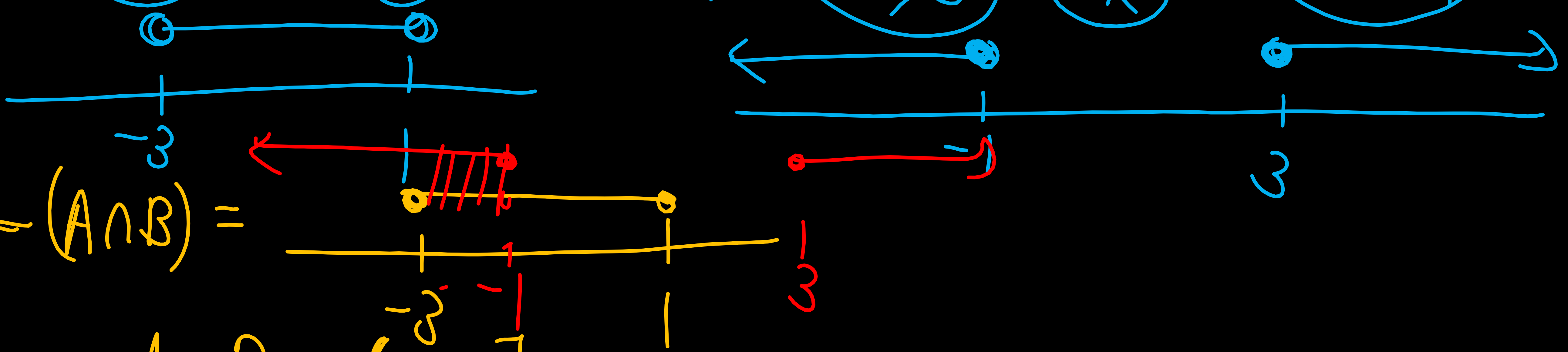
$|\square| \geq 2$

$x - 1 \geq 2$ or $x - 1 \leq -2$

$B: \; x \geq 3$ or $x \leq -1$

$A - B = A - (A \cap B) =$

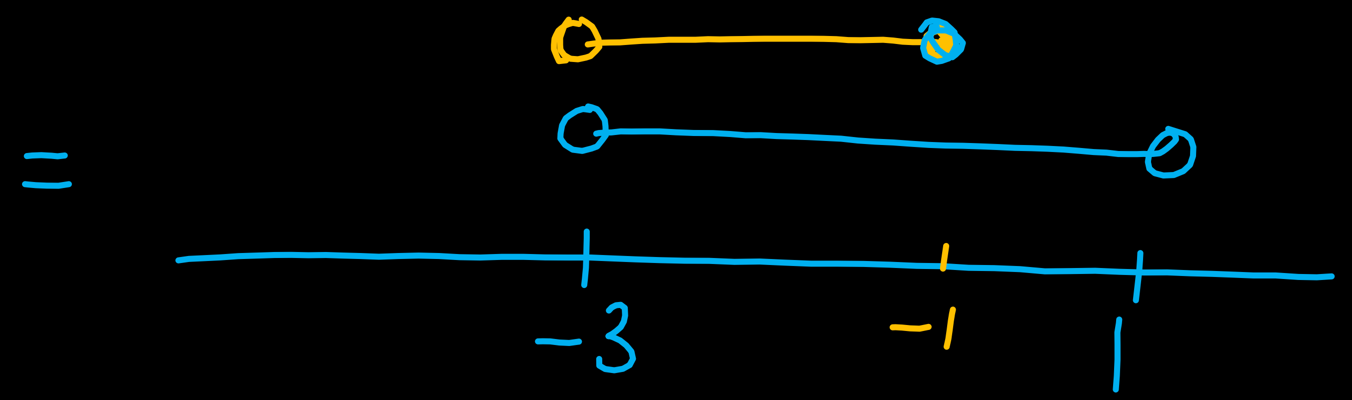
$A - B = (-1, 1]$



$A: \quad -3 < x < 1$

$B: \quad x \in (-\infty, -1] \cup [3, \infty)$

(A) $A - B = A - (A \cap B)$

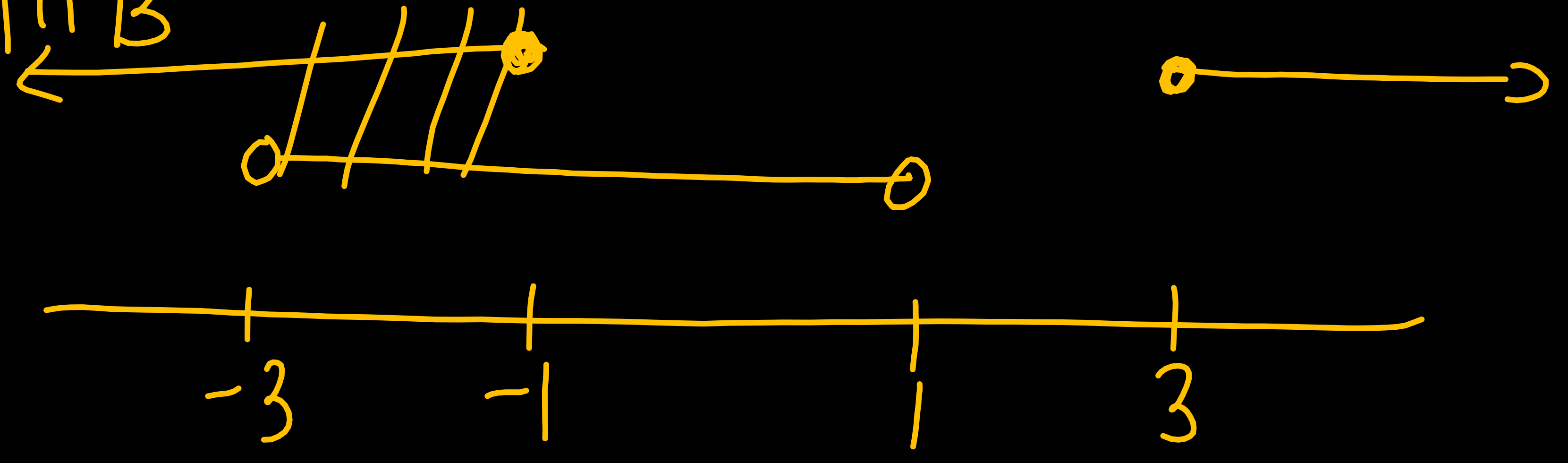


$= (-1, 1)$

(B) $B - A = B - (A \cap B)$

$B - A = (-\infty, -3] \cup [3, \infty)$

$A \cap B$



$A \cap B = (-3, -1]$

$A \cup B : (-\infty, 1) \cup [3, \infty)$

$R - [1, 3)$

Example 9 (JEE Main 2020) Ans:(B)

If $A = \{x \in R : |x| < 2\}$ and $B = \{x \in R : |x - 2| \geq 3\}$; then:

(A) $A \cap B = (-2, -1)$

(B) $B - A = R - (-2, 5)$

(C) $A \cup B = R - (2, 5)$

(D) $A - B = [-1, 2)$

H.W

Example 10 (JEE Main 2022) Ans:3

Let $S = \{x \in [-6,3] - \{-2,2\} : \frac{|x+3|-1}{|x|-2} \geq 0\}$ and $T = \{x \in Z : x^2 - 7|x| + 9 \leq 0\}$. Then the number of elements in $S \cap T$ is:

$T = \{-5, -4, -3, \cancel{-2}, \cancel{2}, 3, \cancel{4}, \cancel{5}\}$

$x \in$ Integres

T: $x^2 - 7|x| + 9 \leq 0$

$|x|^2 - 7|x| + 9 \leq 0$

$|x| = t$ $\quad t^2 - 7t + 9 \leq 0$

$t = \frac{7 \pm \sqrt{49-36}}{2}$

$t = \frac{7 \pm \sqrt{13}}{2}$

$t = \frac{7-\sqrt{13}}{2}$ $\quad t = \frac{7+\sqrt{13}}{2}$

$+\frac{7-\sqrt{13}}{2} \leq t \leq \frac{7+\sqrt{13}}{2} \approx 3.6$

$1.7 \leq |x| \leq 5.3$

$1.7 \leq |x|$ and $|x| \leq 5.3$

$-5.3 \quad -1.7 \quad 1.7 \quad +5.3 \quad -5.3 \quad +5.3$

Example 11 (JEE Main 2023) Ans:(D)

The number of real roots of the equation $x|x| - 5|x+2| + 6 = 0$ is:

(A) 5 (B) 4 (C) 6 (D) 3

$\sqrt{89} = 9.4$

$x|x| - 5|x+2| + 6 = 0$

$x = -2$ $\quad$ $x = 0$

$x(-x) + 5(x+2) + 6 = 0$

$-x^2 + 5x + 16 = 0$

$x^2 - 5x - 16 = 0$ $\quad -16$

$x = \frac{5 \pm \sqrt{25+64}}{2}$

$x = 7.2, -2.2$

$x(-x) - 5(x+2) + 6 = 0$

$-x^2 - 5x - 10 + 6 = 0$

$x^2 + 5x + 4 = 0$ $\quad$ 4, 4, 1

$\boxed{x = -4}$ $\boxed{x = -1}$

Reject

$x(x) - 5(x+2) + 6 = 0$

$x^2 - 5x - 4 = 0$

$x = \frac{5 \pm \sqrt{25+16}}{2}$ $\quad \sqrt{41} \approx 6.4$

$x = 5.7, -0.7$

Reject

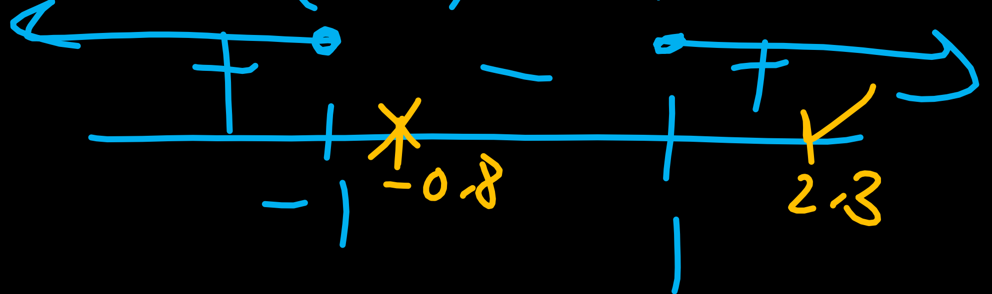
Example 12 (JEE Adv. 2021) Ans:4

For $x \in R$, the number of real roots of the equation $3x^2 - 4|x^2 - 1| + x - 1 = 0$ is:

$$3x^2 - 4\left|(x-1)(x+1)\right| + x - 1 = 0$$

Case I: $x^2 - 1 \geq 0$

$(x-1)(x+1) \geq 0$



$x \in (-\infty, -1] \cup [1, \infty)$

$3x^2 - 4(x^2 - 1) + x - 1 = 0$

$-x^2 + 3 + x = 0$

$x^2 - x - 3 = 0$

$x = \dfrac{1 \pm \sqrt{13}}{2}$

$x = 2.3, -0.8$

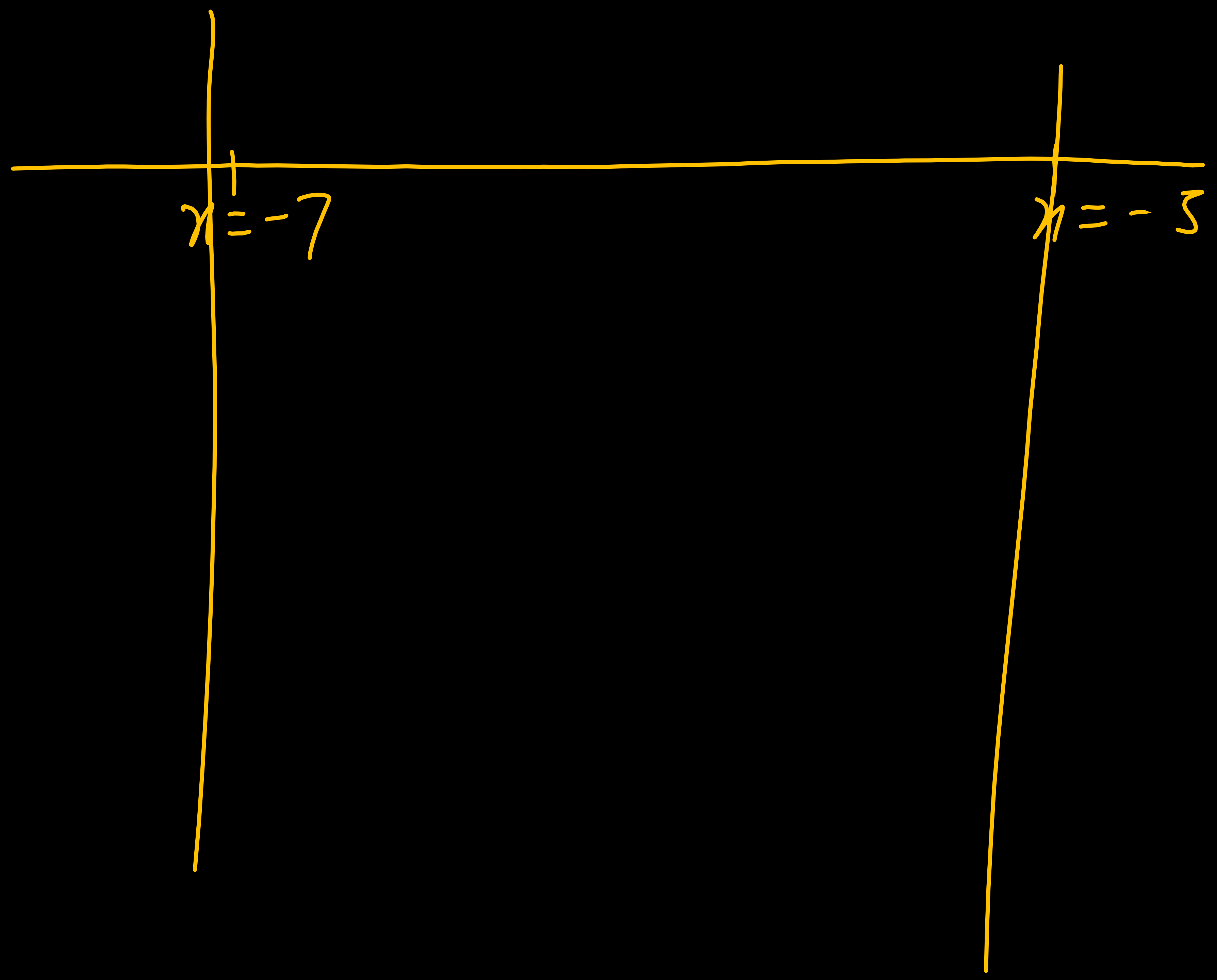
✓ Reject

Case II: $x^2 - 1 < 0$

$x \in (-1, 1)$

Example 13 (JEE Main 2024) Ans:3

The number of real solutions of the equation $x|x+5| + 2|x+7| - 2 = 0$ is:



Example 14 (JEE Main 2024) Ans:2

The number of distinct real roots of the equation $|x+1||x+3| - 4|x+2| + 5 = 0$ is:

$|x||y| = |xy|$

$f(x) = g(x)$

No. of sol?

$y = f(x)$, $y = g(x)$

No. of intersection

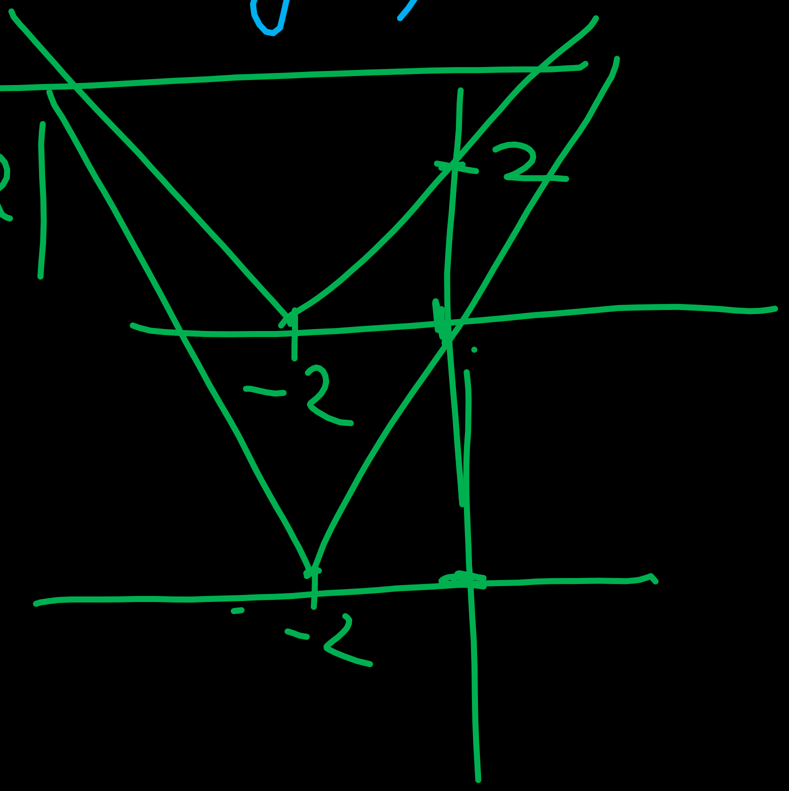
$|(x+1)(x+3)| - 4|x+2| + 5 = 0$

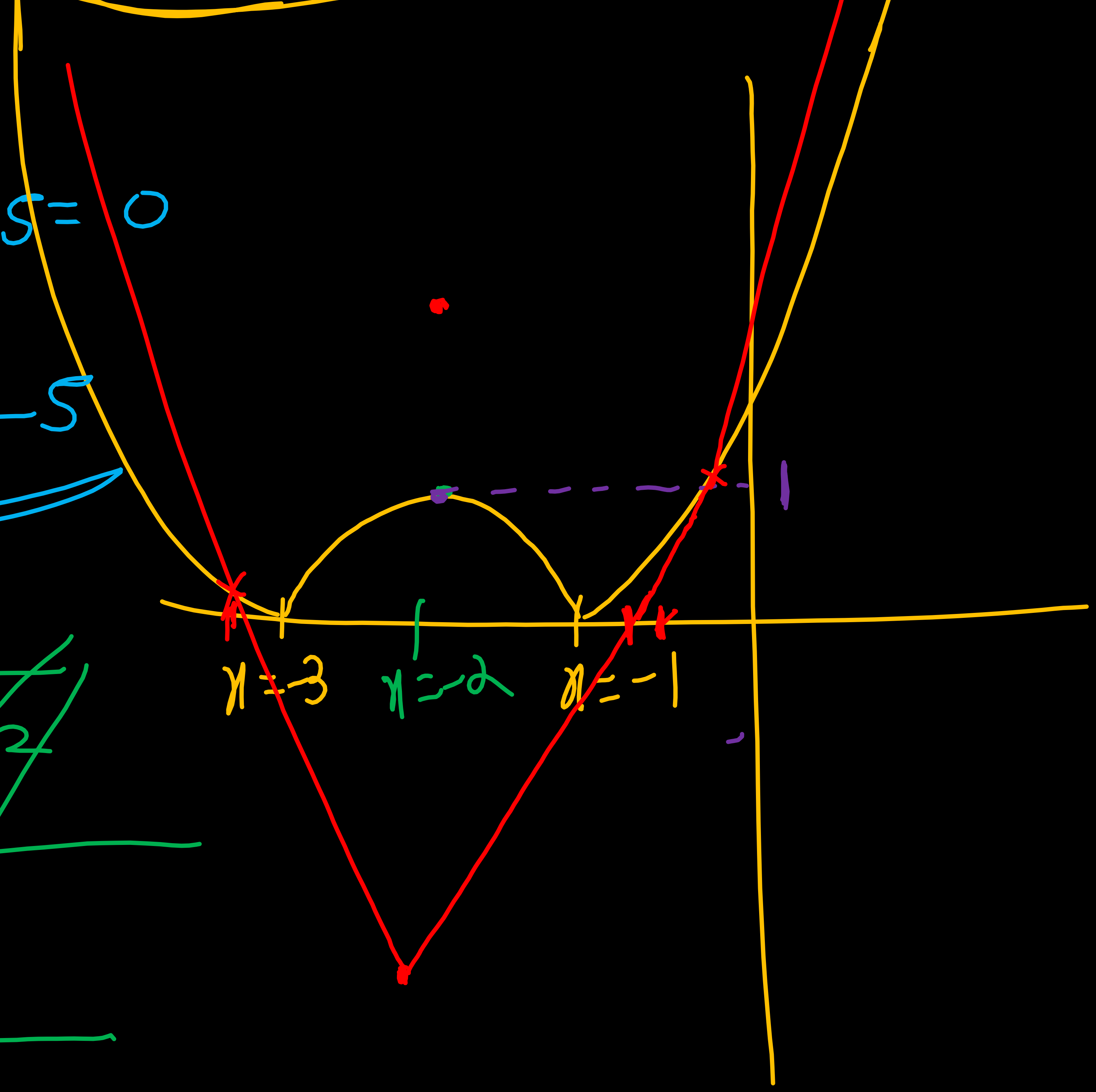
$|x^2+4x+3| = 4|x+2| - 5$

$y = f(x)$ $\quad$ $g(x)$

$y = |x+2|$

$y = 4|x+2|$





Example 15 (JEE Main 2024) Ans:1

The number of real solutions of the equation $x(x^2 + 3|x| + 5|x-1| + 6|x-2|) = 0$ is:

$$x\Big(\underbrace{\underset{\geq 0}{x^2} + 3\underset{\geq 0}{|x|} + 5\underset{\geq 0}{|x-1|} + 6\underset{\geq 0}{|x-2|}}_{>0}\Big) = 0$$

$$x \cdot \blacksquare = 0$$

$$\underline{x = 0}$$

No. of $sol^n = 1$

Example 16 (JEE Main 2019) Ans:(C)

The number of real roots of the equation $5 + |2^x - 1| = 2^x(2^x - 2)$ is:

(A) 2 (B) 3 (C) 1 (D) 4

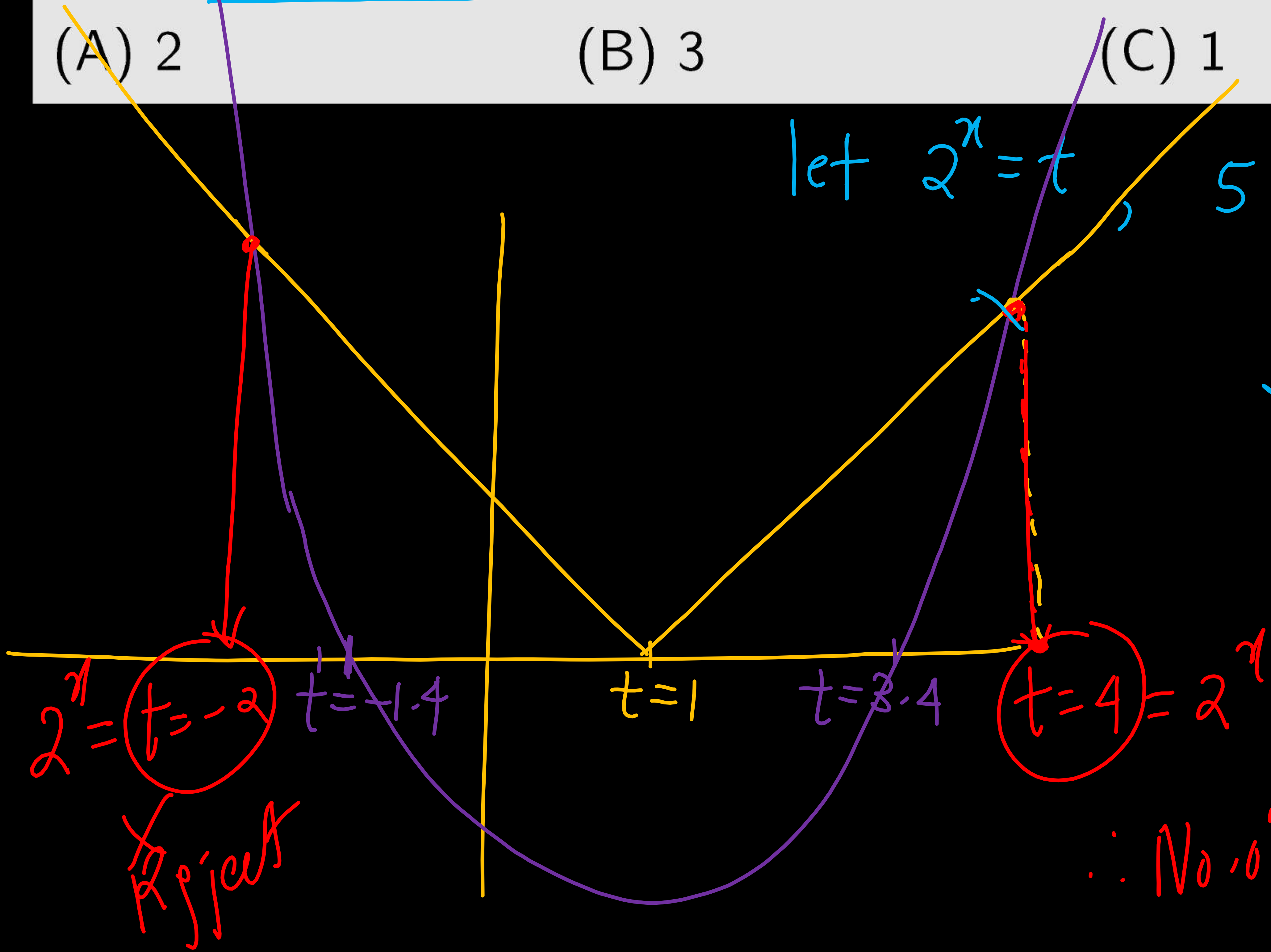
let $2^x = t$, $5 + |t-1| = t(t-2)$

$$\underbrace{|t-1|}_{f(x)} = \underbrace{t^2 - 2t - 5}_{Q.E = g(x)}$$

$g(x) = t^2 - 2t - 5$

$t = \frac{2 \pm \sqrt{4+20}}{2}$

$t = \frac{2 \pm 4.8}{2}$ 3.4, −1.4



$\therefore$ No. of sol^n = 1

Example 17 (JEE Main 2021)

The number of real solutions of the equation, $x^2 - |x| - 12 = 0$ is:

(A) 2 (B) 3 (C) 1 (D) 4

Example 18 (JEE Main 2020) Ans:(B)

The product of the roots of the equation $9x^2 - 18|x| + 5 = 0$ is:

(A) $\frac{5}{9}$ (B) $\frac{25}{81}$ (C) $\frac{5}{27}$ (D) $\frac{25}{9}$

Example 19 (JEE Main 2021) Ans:2

The number of the real roots of the equation $(x+1)^2 + |x-5| = \frac{27}{4}$ is ________.

Case I: $x < 5$

Case II: $x \geq 5$

5

$(x+1)^2 - (x-5) = \frac{27}{4}$

Example 20 (JEE Main 2020) Ans:(B)

Let S be the set of all real roots of the equation, $3^x(3^x - 1) + 2 = |3^x - 1| + |3^x - 2|$, then S:

Number of real roots

(A) contains exactly two elements.

(B) is a singleton.

(C) is an empty set.

(D) contains at least four elements.

$3^x = t$

$$\underbrace{t(t-1)+2}_{\text{OE}=f(x)} = \underbrace{|t-1|+|t-2|}_{g(x)}$$

Example 21 (JEE Main 2019) Ans:(A)

The sum of the solutions of the equation $|\sqrt{x}-2|+\sqrt{x}(\sqrt{x}-4)+2=0$, $(x>0)$ is equal to:

(A) 10 (B) 9 (C) 12 (D) 4

$\sqrt{x}=t>0$

$|t-2|+t(t-4)+2=0$

$|t-2|=-t^2+4t-2$

$|t-2|=-(t^2-4t+2)$

Case I $t\geq 2$

Case II : $t<2$

Example 22 (JEE Main 2023) Ans:(B)

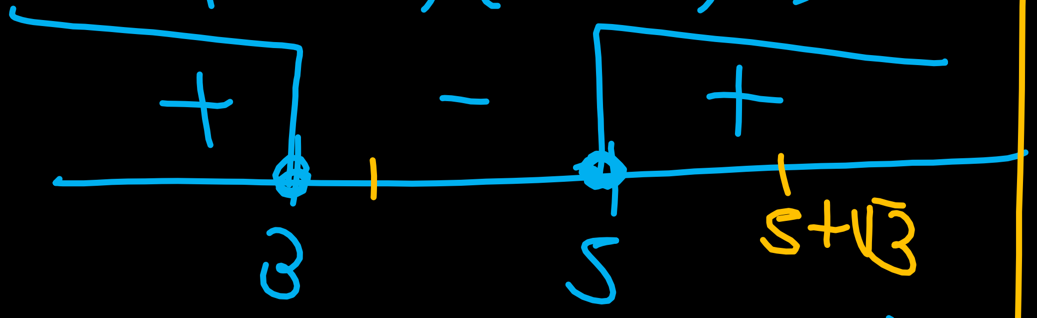
The sum of all the roots of the equation $|x^2 - 8x + 15| - 2x + 7 = 0$ is:

(A) $9 - \sqrt{3}$ (B) $9 + \sqrt{3}$ (C) $11 - \sqrt{3}$ (D) $11 + \sqrt{3}$

$|x^2 - 8x + 15| - 2x + 7 = 0$

Case I: $\boxed{x^2 - 8x + 15 \geq 0}$

$(x-3)(x-5) \geq 0$



$x \in (-\infty, 3] \cup [5, \infty)$

$x^2 - 8x + 15 - 2x + 7 = 0$

$x^2 - 10x + 22 = 0$

$x = \dfrac{10 \pm \sqrt{100 - 88}}{2}$

$x = \dfrac{10 \pm 2\sqrt{3}}{2}$

$x = 5 \pm \sqrt{3}$

$5 + \sqrt{3}$

$5 - \sqrt{3} = 5 - 1.73 = 3.27$ Reject

Case II: $x^2 - 8x + 15 < 0$

$-x^2 + 8x - 15 - 2x + 7 = 0$

$x^2 - 6x + 8 = 0$

$(x-2)(x-4) = 0$

$x = 2$ $\quad$ $x = 4$ ✓

Reject

Sum of roots $= (5 + \sqrt{3}) + (4)$

$= 9 + \sqrt{3}$

Example 23 (JEE Main 2025) Ans:(C)

The sum of the squares of all the roots of the equation $x^2 + |2x - 3| - 4 = 0$, is:

(A) $3(3 - \sqrt{2})$ (B) $6(3 - \sqrt{2})$ (C) $6(2 - \sqrt{2})$ (D) $3(2 - \sqrt{2})$

α, β

Sum of squares roots $= \alpha^2 + \beta^2$

Example 24 (JEE Main 2025) Ans:(B)

The sum of the squares of the roots of $|x-2|^2 + |x-2| - 2 = 0$ and the squares of the roots of $x^2 - 2|x-3| - 5 = 0$, is:

(A) 26 (B) 36 (C) 30 (D) 24

Req $= \alpha^2+\beta^2+\gamma^2+\delta^2$

$= (\alpha+\beta)^2 - 2\alpha\beta + (3)^2 + (1)^2$

$= (-2)^2 - 2(-11) + 9 + 1$

$= 36$

Case I : $x \geq 3$

$x^2 - 2(x-3) - 5 = 0$

$x^2 - 2x + 1 = 0$

$(x-1)^2 = 0$

$x = 1$

Reject

Case II : $x < 3$

$x^2 + 2(x-3) - 5 = 0$

$x^2 + 2x - 11 = 0$ (α, β)

$x = \dfrac{-2 \pm \sqrt{4+44}}{2}$

$x = \dfrac{-2 \pm 4\sqrt{3}}{2}$

$x = -1 \pm 2\sqrt{3}$

$\alpha = -1 + 2\sqrt{3}$ ✓ ($-1 + 2(1.73)$, 2.4)

$\beta = -1 - 2\sqrt{3}$ ✓

$|x-2| = t$

$t^2 + t - 2 = 0$ (-2: $+2$, -1)

$t = -2$ $t = 1$

$|x-2| = -2$ OR $|x-2| = 1$

Reject

$x - 2 = 1$ $x - 2 = -1$

$\gamma = x = 3$ ✓ $\delta = x = 1$ ✓

$$|a+b| \le |a|+|b|$$

$$|a-b| \le |a|+|b|$$

Type

$$\boxed{|a+b| = |a|+|b|}$$

Equality holds,

$$\boxed{a \cdot b \ge 0}$$

$$\boxed{|a-b| = |a|+|b|}$$

Equality holds,

$$\boxed{a \cdot b \le 0}$$

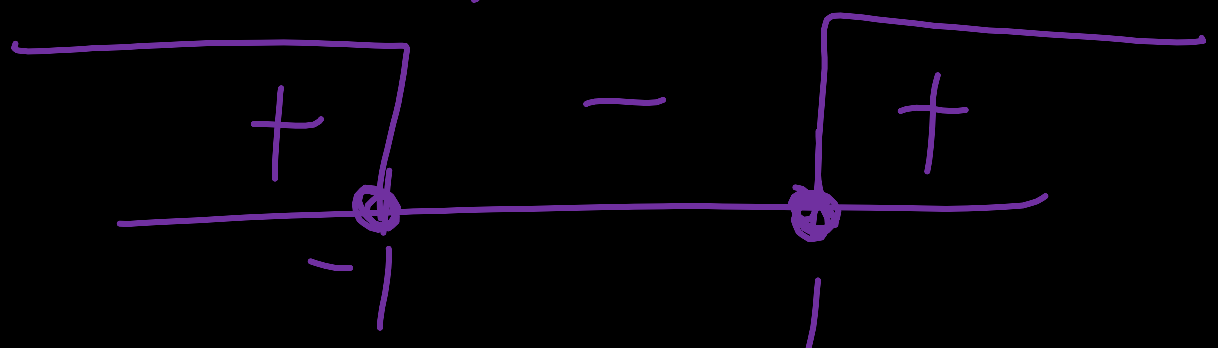
Que ②

(xi) $|x+1| + |x-1| = |2x|$

$$|a| + |b| = |a+b|$$

$$a \cdot b \geq 0$$

$$(x+1)(x-1) \geq 0$$



$$x \in (-\infty, -1] \cup [1, \infty)$$