Basic Mathematics: Section 8

Logarithm

logarithmic Equation / Inequality

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Introduction

36

power | Exponent

$6^2 = 36$ → argument

Base

Exponential form

$\log_6 36 = 2$

log of 36 to the base 6

Logarithmic form

$\log_{\bigcirc} \boxed{}$ → Argument

$\bigcirc$ → Base.

Practice: Find the value

1. $\log_2 4$ $= 2$
2. $\log_3 27$ $= 3$
3. $\log_2 16$ $= 4$
4. $\log_2 \frac{1}{4}$ $= -2$ $\quad 2^x = \frac{1}{4}$
5. $\log_{\frac{1}{2}} 8$ $= -3$ $\quad \left(\frac{1}{2}\right)^x = 8;\ \left(\frac{1}{2}\right)^{-3} = (2^{-1})^{-3} = 8$
6. $\log_{10} 100$ $= 2$
7. $\log_5 1$ $= 0$ $\quad 5^x = 1$
8. $\log_{10} 1$ $= 0$
9. $\log_8 512$ $= 3$
10. $\log_3 81$ $= 4$
11. $\log_{\sqrt{2}} 4$ $= 4$ $\quad (\sqrt{2})^x = 4$
12. $\log_{\sqrt{5}} 125$ $= 6$
13. $\log_5 \frac{1}{5}$ $= -1$ $\quad 5^x = \frac{1}{5}$
14. $\log_{\sqrt{3}} \frac{1}{3}$ $= -2$ $\quad (\sqrt{3})^x = \frac{1}{3}$
15. $\log_3 \frac{1}{9}$ $= -2$ $\quad \sqrt{2} - 1 = \frac{1}{\sqrt{2}+1}$
16. $\log_{\sqrt{2}+1}(\sqrt{2} - 1)$ $= -1$ $\quad (\sqrt{2}-1)(\sqrt{2}+1) = 1$

Example 17

$$\log_3(27\sqrt{3})$$

मालीक

$\log_3 \boxed{27\sqrt{3}} = x$

नौकर

$27\sqrt{3} = (3)^x$

$3^3 \cdot 3^{1/2} = 3^x$

$3^{7/2} = 3^x$

$x = 7/2$

Example 18

$$\log_{\sqrt{8}} 16$$

$\log_{\sqrt{8}} 16 = x$

$16 = (\sqrt{8})^x$

$2^4 = (2^{3/2})^x$

$4 = \frac{3x}{2}$

$x = 8/3$

Some Common Logarithm Results

Result 1

$\log_{\frac{1}{a}} a = -1$

$$\log_{\frac{1}{a}} a = -1$$

e.g $\cancel{\log}_{\frac{1}{5}} 5 = x$

$5 = \left(\frac{1}{5}\right)^x$

$5^1 = 5^{-x}$

$x = -1$

e.g $\log_{\frac{1}{\sqrt{2}}} \sqrt{2} = -1$

Result 2

$\log_a \left(\frac{1}{a}\right) = -1$

$$\log_a \frac{1}{a} = -1$$

e.g $\log_5 \frac{1}{5} = -1$

e.g $\log_{\sqrt{7}} \frac{1}{\sqrt{7}} = -1$

Result 3

$$\log_a 1 = 0$$

e.g $\log_2 1 = 0$ $\quad 2^x = 1$

$\log_{\sqrt{10}} 1 = 0$ $\quad (\sqrt{10})^x = 1$

$\log_{\square} 1 = 0$

Restriction of Logarithmic Function (Domain)

$$f(x) = \log_{\circ} \square$$

i) $\square > 0$

ii) $\circ > 0$

iii) $\circ \neq 1$

tl

Li

kg

M

$\cancel{\log}_{-2} 2 = x$

$2 = (-2)^x$

$\cancel{\log}_{1} 2 = x$

$2 = 1^x$

P

4 5 10

20

$\frac{0/7}{20}$

$\frac{8/9}{20}$

100%

2500

Result 4: Base and Argument are Equal

$$\boxed{\log_a a = 1} \quad (\text{where } a > 0, a \neq 1)$$

In exponential form, this is equivalent to $a^1 = a$.

e.g $\log_5 5 = 1$

$\log_{\sqrt[3]{7}} \sqrt[3]{7} = 1$

$\log_{81} 81 = 1$

Result 5: Power Rule for Argument

$\log_a \boxed{m}^n = n \cdot \log_a m$

$$\log_a(m^n) = n \cdot \log_a m$$

e.g., $\log_5 25 = 2$

$\log_5 5^2 = 2 \cdot \log_5 5 = 2$

e.g $\log_3 81 = \log_3 \boxed{3}^4 = 4 \cdot \log_3 3 = 4$

Result 6: Power Rule for Base

$$\log_{a^\beta} m = \frac{1}{\beta} \log_a m$$

$\log_{a^\beta} m = \frac{1}{\beta} \log_a m$

e.g $\log_{\sqrt{8}} \sqrt[3]{16} = \log_{\boxed{2}^{3/2}} \boxed{2}^{4/3}$

$= \frac{4}{3} \cdot \frac{2}{3} \log_2 2$

$= \frac{8}{9}$

$a^m \cdot a^n = a^{m+n}$

Result 7: Product Rule

$$\log_a m + \log_a n = \log_a(mn)$$

Result 8: Quotient Rule

$$\log_a m - \log_a n = \log_a\left(\frac{m}{n}\right)$$

e.g. $\log_{10} 5 + \log_{10} 20 = \log_{10}(5 \times 20) = \log_{10} 100 = 2$

e.g. $\log_3 2 + \log_3 4 - \log_3 10 + \log_3 20 - \log_3 8$

$\log_3\left(\frac{\cancel{2 \times 4} \times 20}{10 \times \cancel{8}}\right) = \log_3 2$

$\log_2(20) = \log_2(2 \times 10)$

$= \log_2 2 + \log_2 10$

$\log_2\left(\frac{25}{4}\right) = \log_2 25 - \log_2 4$

$\log_a m \cdot \log_a n \neq$ ✗

$\log_a(m+n) =$ ✗

Result 9: Base Change Theorem

$$\log_b a = \frac{\log_c a}{\log_c b}$$

$$\log_{\boxed{b}} \boxed{a} = \frac{\log_c \boxed{a}}{\log_c \boxed{b}}$$

$$\log_3 12 = \frac{\log_2 12}{\log_2 3}$$

$$\frac{\log_{10} 5}{\log_{10} 25} = \log_{25} 5 = \frac{1}{2}$$

$$(25)^{x} = 5$$

Result 10: Reciprocal Property

$$\log_b a = \frac{1}{\log_a b}$$

$$\boxed{\log_{\boxed{b}} \boxed{a} = \frac{1}{\log_a b}}$$

Proof:

$$\log_b a \cdot \log_a b$$

$$= \frac{\cancel{\log a}}{\cancel{\log b}} \cdot \frac{\cancel{\log b}}{\cancel{\log a}}$$

$$= 1$$

Result 11:

$$a^{\log_a N} = N$$

$$a^{\log_a N} = N$$

e.g $3^{\log_3 5} = 5$

$25^{\log_5 7} = (5^2)^{\log_5 7} = 5^{2 \times \log_5 7} = 5^{\log_5 7^2} = 7^2$

Example 1

Find the value of:

$$\sum_{n=1}^{1023} \log_2\left(1+\frac{1}{n}\right)$$

$$\sum_{n=1}^{1023} \log_2\left(1+\frac{1}{n}\right) = \log_2\left(1+\frac{1}{1}\right) + \log_2\left(1+\frac{1}{2}\right) + \log_2\left(1+\frac{1}{3}\right) + \dots + \log_2\left(1+\frac{1}{1023}\right)$$

$$= \log_2\left(\frac{2}{1}\right) + \log_2\left(\frac{3}{2}\right) + \log_2\left(\frac{4}{3}\right) + \log_2\left(\frac{5}{4}\right) + \dots + \log_2\left(\frac{1024}{1023}\right)$$

$$= \log_2\left(\left(\frac{\cancel{2}}{1}\right) \times \frac{\cancel{3}}{\cancel{2}} \times \frac{\cancel{4}}{\cancel{3}} \cdot \frac{\cancel{5}}{\cancel{4}} \cdots \frac{1024}{\cancel{1023}}\right)$$

$$= \log_2 1024$$

$$= 10$$

Example 2

Find the value of:

$$\log_4(\log_2(\underline{\log_{25} 625}))$$

$\log_{\boxed{2}} \boxed{8} = 3$ → argument

→ Base

$2^x = 8$

$\log_{\boxed{2}} \boxed{8} = x$

$8 = 2^x$

$x = 3$

$= \log_4(\log_2 2)$

$= \log_4 1 \quad ; 4^x = 1$

$= 0$

Example 3

If $\log_2(\log_2(\log_3 x)) = 0 = \log_2(\log_3(\log_2 y))$, then find $x + y$.

$$\cancel{\log_2}\left(\log_2\left(\log_3 x\right)\right) = 0$$

$$\cancel{\log_2}\left(\log_3 x\right) = 2^0$$

$$\cancel{\log_3} x = 2^1$$

$$x = 3^2$$

$$x = 9$$

$$\cancel{\log_2}\left(\log_3\left(\log_2 y\right)\right) = 0$$

$$\cancel{\log_3} \log_2 y = 1$$

$$\cancel{\log_2} y = 3^1$$

$$y = 2^3$$

$$y = 8$$

$$x + y = 9 + 8 = 17$$

Example 4

Find the value of:

$$\log_{10}\left(\frac{75}{16}\right) - 2\log_{10}\left(\frac{5}{9}\right) + \log_{10}\left(\frac{32}{243}\right)$$

$3^1 = 3$

$3^2 = 9$

$3^3 = 27$

$3^4 = 81$ $\quad 3^5 = 243$

$$\left(\log_{10} 75 - \log_{10} 16\right) - 2\left(\log_{10} 5 - \log_{10} 9\right) + \log_{10} 32 - \log_{10} 243$$

$$\left(\log_{10} 5^2 \cdot 3 - \log_{10} 2^4\right) - 2\left(\log_{10} 5 - \log_{10} 3^2\right) + \left(\log_{10} 2^5 - \log_{10} 3^5\right)$$

$$\cancel{2 \cdot \log_{10} 5} + \cancel{\log_{10} 3} - 4 \cdot \log_{10} 2 - \cancel{2\log_{10} 5} + \cancel{4 \cdot \log_{10} 3} + 5 \cdot \log_{10} 2 - \cancel{5 \cdot \log_{10} 3}$$

$$\boxed{\log_{10} 2}$$

Example 5

Find the value of: $\log m + \log n = \log mn$

$$\log \tan 1^\circ + \log \tan 2^\circ + \log \tan 3^\circ + \cdots + \log \tan 89^\circ$$

$$\log\left(\tan 1^\circ \cdot \tan 2^\circ \cdot \tan 3^\circ \cdots \tan 89^\circ\right)$$

$$\log\left[(\tan 1^\circ \cdot \tan 89^\circ)(\tan 2^\circ \cdot \tan 88^\circ)(\tan 3^\circ \cdot \tan 87^\circ) \cdots (\tan 44^\circ \cdot \tan 46^\circ)(\tan 45^\circ)\right]$$

$$\log\left[(\tan 1^\circ \cdot \cot 1^\circ)(\tan 2^\circ \cdot \cot 2^\circ)(\tan 3^\circ \cdot \cot 3^\circ) \cdots (\tan 44^\circ \cdot \cot 44^\circ)(1)\right]$$

$$= \log 1 = 0$$

$\tan(90 - \theta) = \cot\theta$

$\tan(89) = \tan(90 - 1^\circ) = \cot 1^\circ$

$\tan 88 = \cot 2^\circ$

$\tan 30 = \cot 60$

$\frac{1}{2} = \sin 30 = \cos 60$

Example 6

Find the value of:

$$\log\tan 1^\circ \cdot \log\tan 2^\circ \cdot \log\tan 3^\circ \cdots \log\tan 89^\circ$$

$\cancel{\log m \cdot \log n = \log(m+n)}$

$\log m + \log n = \log mn$

$(\log\tan 44^\circ)(\log \underline{\tan 45^\circ})(\log\tan 46^\circ)$

$\log \tan 45^\circ = \log 1 = \text{zero}$

Ans: zero

Example 7

Find the value of:

$$\frac{1}{9}^{\log_{27} 8}$$

$$\left(\frac{1}{9}\right)^{\log_{27} 8} = \left(3^{-2}\right)^{\log_{3^3} 2^3}$$

$$= \left(3^{-2}\right)^{\cancel{\frac{3}{3}} \log_3 2}$$

$$= \left(3^{-2}\right)^{\log_3 2} = 3^{-2 \cdot \log_3 2} = 3^{\log_3 2^{-2}} = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$$

$$\boxed{a^{\log_a N} = N}$$

$$(a^m)^n = a^{m \times n}$$

Example 8

Find the value of:

$$\sqrt{2}^{-3\log_4 5}$$

$$\left(\sqrt{2}\right)^{-3\log_4 5} = \left(\left(2\right)^{\frac{1}{2}}\right)^{\left(-3\,\frac{1}{2}\log_2 5\right)}$$

$$= 2^{\frac{1}{2}\cdot(-3)\cdot\left(\frac{1}{2}\right)\cdot\log_2 5}$$

$$= 2^{\log_2 5^{\left(\frac{-3}{4}\right)}}$$

$$= (5)^{-3/4}$$

$$\boxed{a^{\log_a N} = N}$$

Example 9

Solve for x in the equation:

$$2^{\log_2 e^{\ln 5^{\log_7 7^{\log_3 3^{\log_3 (8x-3)}}}}} = 13$$

$$a^{\log_a N} = N$$

$$2^{\log_2 e^{\ln 5^{\log_5 7^{\log_7 3^{\log_3 (8x-3)}}}}} = 13$$

$8x - 3 = 13$

$8x = 16$

$x = 2$

$\log_{10} \square \longrightarrow$ Base 10

Common log

$\log_e \square = \ln \square \longrightarrow$ Base $= e$

Natural log

$e = 2.718$

Example 10

Find the value of n that satisfies the equation:

$$\log_2 3 \cdot \log_3 4 \cdot \log_4 5 \cdots \log_n(n+1) = 10$$

Base-Change Thm

$$\boxed{\log_b a = \frac{\log_{\square} a}{\log_{\square} b}}$$

$$\frac{\cancel{\log 3}}{\log 2} \cdot \frac{\cancel{\log 4}}{\cancel{\log 3}} \cdot \frac{\cancel{\log 5}}{\cancel{\log 4}} \cdots \frac{\log(n+1)}{\cancel{\log n}} = 10$$

$$\frac{\log(n+1)}{\log 2} = 10$$

$$\log_2(n+1) = 10$$

$$n+1 = 2^{10}$$

$$n = 1024 - 1$$

$$n = 1023$$

Example 11

Find the value of the expression:

$$\frac{1}{\log_{xy} xyz} + \frac{1}{\log_{yz} xyz} + \frac{1}{\log_{zx} xyz}$$

$$\boxed{\log_{\boxed{b}} \boxed{a} = \frac{1}{\log_{a} b}}$$

$$\log_{xyz}(x \cdot y) + \log_{xyz} yz + \log_{xyz} zx =$$

$$\log_{xyz} (xy)(yz)(xz)$$

$$\log_{xyz} (xyz)^2 = 2 \cdot \cancel{\log_{xyz} xyz}^{\,1} = 2$$

Example 12

If $P = 2026!$, find the value of N where:

$$N = \sum_{r=2}^{2026} \frac{1}{\log_r P}$$

$2! = 2 \times 1$

$3! = 3 \times 2 \times 1$

$5! = 5 \times 4 \times 3 \times 2 \times 1$

$$N = \frac{1}{\log_2 P} + \frac{1}{\log_3 P} + \frac{1}{\log_4 P} + \cdots + \frac{1}{\log_{2026} P}$$

$$= \log_P 2 + \log_P 3 + \log_P 4 + \cdots + \log_P 2026$$

$$= \log_P (2 \cdot 3 \cdot 4 \cdot 5 \cdot \ldots \cdot 2026)$$

$$= \log_P 2026! = 1$$

Result 12: Base-Argument Interchange Property

$$a^{\log_c b} = b^{\log_c a}$$

Example

$$5^{\log_2 9} = 9^{\log_2 5}$$

$$a^{\log_c b} = b^{\log_c a}$$

$$= a^{\left(\frac{\log_a b}{\log_a c}\right)} = a^{(\log_a b)\left(\frac{1}{\log_a c}\right)} = \left(a^{\log_a b}\right)^{\frac{1}{\log_a c}} = b^{\frac{1}{\log_a c}} = b^{\log_c a}$$

$$a^{m \times n} = (a^m)^n$$

Example 13

Find the value of:

$$7^{\log_3 5} + 3^{\log_5 7} - 5^{\log_3 7} - 7^{\log_5 3}$$

$$\cancel{7^{\log_3 5}} + \cancel{7^{\log_5 3}} - \cancel{7^{\log_3 5}} - \cancel{7^{\log_5 3}}$$

0

Example 14

Find x in the equation:

$$2^{\log_3 x} + 8 = 3x^{\log_9 4}$$

Take log on both sides:

$\log_4 4 = \log_4 x^{\log_3 2}$

$1 = \log_3 2 \cdot \log_4 x$

$\frac{1}{\log_3 2} = \log_4 x$

$x^{\log_3 2} + 8 = 3 \cdot x^{\log_{3^2} 2^2}$

$x^{\log_3 2} + 8 = 3 \cdot x^{\log_3 2}$

$\cancel{8} = \cancel{2} \cdot x^{\log_3 2}$

$4 = x^{\log_3 2}$

$\log_2 3 = \log_4 x$

$\log_2 3 = \frac{1}{2} \log_2 x$

$\cancel{\log_2} 3 = \cancel{\log_2} (x)^{1/2}$

$\sqrt{x} = 3 \Rightarrow x = 9$

Logarithmic Equation

Part-01 : Properties

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Example 1

Solve the equation:

$$2^{\log_2(x^2)} - 3x - 4 = 0$$

$$x^2 - 3x - 4 = 0$$

Example 2

Solve for x:

$$\frac{\log_2(9-2^x)}{(3-x)} = 1$$

$$\cancel{\log_2}(9-2^x) = (3-x)$$

$$9-2^x = 2^{(3-x)}$$

$$a^{m-n} = \frac{a^m}{a^n}$$

$$9-2^x = \frac{2^3}{2^x}$$

let $2^x = t$

$$9-t = \frac{8}{t}$$

$$9t - t^2 = 8$$

$$t^2 - 9t + 8 = 0$$

$$8 \to -8,\ -1$$

$t=8$	$t=1$
$2^x = 2^3$	$2^x = 1$
$x=3$	$x=0$

Example 3

Solve for x:

$$\cancel{\log_4}(2\log_3(1+\log_2(1+3\log_3 x))) = \frac{1}{2}$$

$$\cancel{2}\cdot \cancel{\log_3}\left(1+\log_2\left(1+3\cdot\log_3 x\right)\right) = \cancel{2}$$

$$1+\log_2\left(1+3\cdot\log_3 x\right) = 3^1$$

$$\cancel{\log_2}\left(1+3\cdot\log_3 x\right) = 2$$

$$1+3\cdot\log_3 x = 4$$

$$\cancel{3}\cdot\log_3 x = \cancel{3}$$

$$\cancel{\log_3} x = 1$$

$$x = 3$$

Example 4

Solve for x:

$$\log_7(2^x - 1) + \log_7(2^x - 7) = 1$$

$\log m + \log n = \log mn$

$\cancel{\log_7}(2^x-1)(2^x-7) = 1$

$(2^x-1)(2^x-7) = 7^1$

let $2^x = t$,

$(t-1)(t-7) = 7$

$t^2 - 8t + \cancel{7} = \cancel{7}$

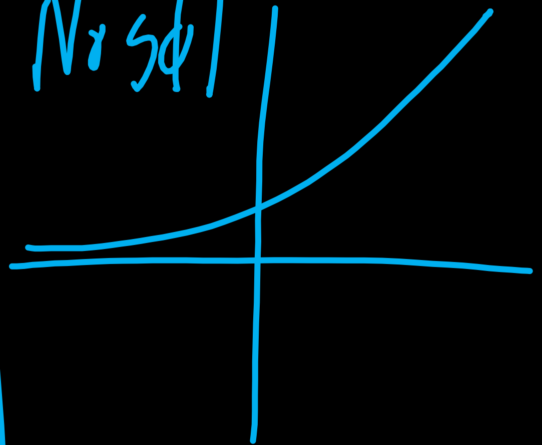
$t^2 - 8t = 0$

$t(t-8) = 0$

$t = 0$ $\quad$ $t = 8$

$2^x = 0$ $\quad$ $2^x = 8$

No sol'n $\quad$ $x = 3$



Example 5

Solve for x:

$$\log_{x-1} 4 = 1 + \log_2(x-1)$$

$$2 \cdot \log_{(x-1)} 2 = 1 + \log_2(x-1)$$

$$\frac{2}{\log_2(x-1)} = 1 + \log_2(x-1)$$

let $\log_2(x-1) = t$,

Example 6

Solve for x:

$$(x+1)^{\log_{10}(x+1)} = 100(x+1)$$

take log on both sides.

$$\log_{10}(x+1)^{\log_{10}(x+1)} = \log_{10} 100(x+1)$$

$$\left(\log_{10}(x+1)\right) \cdot \left(\log_{10}(x+1)\right) = \underbrace{\log_{10} 100}_{2} + \log_{10}(x+1)$$

let $\log_{10}(x+1) = t$

$$t^2 = 2 + t$$

Example 7

Solve for x: $3^{\log_3 \log\sqrt{x}}$ $\cancel{3^{\log_3 \sqrt{x}}} - \log x + \log^2 x - 3 = 0$

$\log^2 x = (\log x)^2$

$\log x^2 =$

$\log\sqrt{x} - \log x + (\log x)^2 - 3 = 0$

$\log x = t$

$\frac{1}{2}t - t + t^2 - 3 = 0$

$t^2 - \frac{t}{2} - 3 = 0$

$2t^2 - t - 6 = 0 \quad -12$

$2t^2 - 4t + 3t - 6 = 0 \quad -4 \quad +3$

$2t(t-2) + 3(t-2) = 0$

$t = 2 \qquad t = \frac{-3}{2}$

$\log_{10} x = 2 \qquad \log_{10} x = \frac{-3}{2}$

$x = 10^2 \qquad x = 10^{-3/2}$

Example 8

Solve for x:

$$\log x - \frac{1}{2}\log\left(x-\frac{1}{2}\right) = \log\left(x+\frac{1}{2}\right) - \frac{1}{2}\log\left(x+\frac{1}{8}\right)$$

$\log m - \log n = \log m/n$

$$\log x - \log\left(x-\frac{1}{2}\right)^{1/2} = \log\left(x+\frac{1}{2}\right) - \log\left(x+\frac{1}{8}\right)^{1/2}$$

$$\cancel{\log}\left(\frac{x}{\sqrt{x-\frac{1}{2}}}\right) = \cancel{\log}\left(\frac{x+\frac{1}{2}}{\sqrt{x+\frac{1}{8}}}\right)$$

$$\frac{x^2}{x-\frac{1}{2}} = \frac{x^2+\frac{1}{4}+x}{x+\frac{1}{8}}$$

Example: 9 [JEE Main 2023] Ans:(D)

If the solution of the equation → Find value of x

$$\log_{\cos x}(\cot x) + 4\log_{\sin x}(\tan x) = 1, \quad x \in \left(0, \frac{\pi}{2}\right)$$

I st quad.

is $\boxed{\sin^{-1}\left(\frac{\alpha+\sqrt{\beta}}{2}\right)}$, where α, β are integers, then $\alpha+\beta$ is equal to:

(A) 3 (B) 5 (C) 6 (D) 4

$$\log_{\cos x}\left(\frac{\cos x}{\sin x}\right) + 4\cdot\log_{\sin x}\left(\frac{\sin x}{\cos x}\right) = 1$$

$$\cancel{\log_{\cos x}\cos x} - \log_{\cos x}\sin x + 4\left(\cancel{\log_{\sin x}\sin x} - \log_{\sin x}\cos x\right) = 1$$

$1 \quad 1$

let $\log_{\cos x}\sin x = t$

$$\cancel{1} - t + 4\left(1 - \frac{1}{t}\right) = \cancel{1}$$

$$-t^2 + 4t - 4 = 0$$

$$t^2 - 4t + 4 = 0$$

$$(t-2)^2 = 0$$

$$t = 2$$

$$\cancel{\log_{\cos x}}\sin x = 2$$

$$\sin x = (\cos x)^2$$

$$\sin x = \cos^2 x$$

$$\downarrow$$

$$\sin x = 1 - \sin^2 x$$

$$\sin^2 x + \sin x - 1 = 0$$

$$\sin x = \frac{-1 \pm \sqrt{1 - 4(1)(-1)}}{2}$$

$$\sin x = \frac{-1 \pm \sqrt{5}}{2}$$

~~Reject~~

$$\sin x = \frac{-1 - \sqrt{5}}{2} \qquad \boxed{\sin x = \frac{-1 + \sqrt{5}}{2}}$$

$$\sin x = \frac{-1 + \sqrt{5}}{2}$$

$$x = \sin^{-1}\left(\frac{-1 + \sqrt{5}}{2}\right) = \sin^{-1}\left(\frac{\alpha + \sqrt{\beta}}{2}\right)$$

$$\alpha = -1 \qquad \beta = 5$$

$$\alpha + \beta = -1 + 5 = 4 \;//$$

Example: 10 [JEE MAIN 2021] Ans:(B)

The number of solution(s) of $\log_4(x-1) = \log_2(x-3)$ is:

(A) 3 (B) 1 (C) 2 (D) 0

$$\log_4(x-1) = \log_2(x-3)$$

$$\frac{1}{2}\log_2(x-1) = \log_2(x-3)$$

$$\cancel{\log_2}(x-1) = \cancel{\log_2}(x-3)^2$$

$$(x-1) = (x-3)^2$$

$x=5$ ✓ & $x=2$ ✗ Reject

Check answer with domain

Example: 11 [JEE MAIN 2021] Ans:1

Find the number of solutions of the equation, for $x > 0$:

$$\log_{(x+1)}(2x^2+7x+5)+\log_{(2x+5)}(x+1)^2-4=0$$

$\cancel{x=-4}$, ± 2

$x=-2$ Reject

$x=2$

Chalaki

$2x^2+7x+5=0$ 10, 5, 2

$2x^2+2x+5x+5=0$

$2x(x+1)+5(x+1)=0$

$(x+1)(2x+5)$

$\log_{(x+1)}(x+1)(2x+5)+\log_{(2x+5)}(x+1)^2-4=0$

$\log_{(x+1)}(x+1)+\log_{(x+1)}(2x+5)+2\log_{(2x+5)}(x+1)-4=0$

1

let $\log_{(x+1)}(2x+5)=t$

$1+t+\frac{2}{t}-4=0$

$t^2-3t+2=0 \Rightarrow t=1$ & $t=2$

Example: 12 [JEE Main 2021] Ans:(D)

The sum of the roots of the equation,

$x_1 + x_2 + x_3 + \cdots$

$$x + 1 - 2\log_2(3 + 2^x) + 2\log_4(10 - 2^{-x}) = 0,$$

x_1, x_2, x_3, ⋮

is:

(A) $\log_2 14$ (B) $\log_2 12$ (C) $\log_2 13$ (D) $\log_2 11$ ✓

let $2^x = t$

$\log_2 2^x = \log_2 t$

$x = \log_2 t$

$\log_2 t + 1 - 2\log_2(3+t) + \frac{\cancel{2}}{\cancel{2}}\log_2(10 - \frac{1}{t}) = 0$

$\log_2 t + \log_2 2 - \log_2 (3+t)^2 + \log_2\left(\frac{10t-1}{t}\right) = 0$

$\cancel{\log_2}\left(\frac{\cancel{t} \times 2}{(t+3)^2} \times \left(\frac{10t-1}{\cancel{t}}\right)\right) = 0$

$$\frac{20t-2}{(t+3)^2} = 1$$

$$20t-2 = t^2+6t+9$$

$$t^2-14t+11=0$$

$$t_1 = 2^{x_1}$$

$$t_2 = 2^{x_2}$$

$$t_1+t_2 = 14$$

$$t_1 t_2 = 11$$

$$t_1 t_2 = 11$$

$$2^{x_1} \cdot 2^{x_2} = 11$$

$$2^{x_1+x_2} = 11$$

$$\log_2 2^{x_1+x_2} = \log_2 11$$

$$x_1+x_2 = \log_2 11$$

Example: 13 [JEE Main 2021] Ans:(B)

If for $x \in (0, \frac{\pi}{2})$, $\log_{10}\sin x + \log_{10}\cos x = -1$ and $\log_{10}(\sin x + \cos x) = \frac{1}{2}(\log_{10} n - 1)$, $n > 0$, then the value of n is equal to:

(A) 20 (B) 12 (C) 9 (D) 16

$\log_{10}\sin x + \log_{10}\cos x = -1$

$\cancel{\log_{10}} \sin x . \cos x = -1$

$\sin x . \cos x = \frac{1}{10}$

$\log_{10}(\sin x + \cos x) = \frac{1}{2}(\log_{10} n - 1)$

$2 . \log_{10}(\sin x + \cos x) = \log_{10} n - \log_{10} 10$

$\cancel{\log_{10}}(\sin x + \cos x)^2 = \cancel{\log_{10}}\left(\frac{n}{10}\right)$

$1 + 2\sin x . \cos x = \frac{n}{10}$

$1 + \frac{2}{10} = \frac{n}{10} \Rightarrow n = 12$

Example:14 [JEE Advanced 2022] Ans:1

The product of all positive real values of x satisfying the equation

Req $= x_1 x_2 x_3 \dots$

$$x^{\left(16(\log_5 x)^3 - 68\log_5 x\right)} = 5^{-16}$$

x_1, x_2, x_3, ...

is $\underline{\quad 1 \quad}$.

take $\log_5$ on both sides.

$$\log_5 \boxed{x} \left(16.(\log_5 x)^3 - 68\log_5 x\right) = \log_5 5^{-16}$$

let $\log_5 x_1 = t_1$

$$(16t^3 - 68t)\,t = -16$$

$$16t^4 - 68t^2 + 16 = 0$$

$$4t^4 - 17t^2 + 4 = 0$$

t_1, t_2, t_3, t_4

$t_1 + t_2 + t_3 + t_4 = 0 \rightarrow$ coeff of $t^3 = 0$

$$\log_5 x_1 + \log_5 x_2 + \log_5 x_3 + \log_5 x_4 = 0$$

$$\log_5 x_1 x_2 x_3 x_4 = 0 \Rightarrow x_1 x_2 x_3 x_4 = 1$$

Theory of Eqⁿ (ToE)

$+ \quad - \quad +$

$$ax^2+bx+c=0 \qquad \alpha, \ \beta$$

$$\alpha+\beta=\frac{-b}{a} \qquad \alpha\beta=\frac{c}{a}$$

$+ \quad - \quad + \quad -$

$$ax^3+bx^2+cx+d=0 \qquad \alpha, \ \beta, \ \gamma$$

$$\alpha+\beta+\gamma=\frac{-b}{a}$$

$$\alpha\beta+\beta\gamma+\alpha\gamma=\frac{c}{a}$$

$$\alpha\beta\gamma=\frac{-d}{a}$$

$+ \quad - \quad + \quad - \quad +$

$$ax^4+bx^3+cx^2+dx+e=0 \qquad x_1, \ x_2, \ x_3, \ x_4$$

$$x_1+x_2+x_3+x_4=\frac{-b}{a}$$

$$x_1x_2+x_1x_3+\ldots=\sum x_i x_j=\frac{c}{a}$$

$$\sum x_1x_2x_3=\frac{-d}{a}$$

$$x_1x_2x_3x_4=\frac{e}{a}$$

Example: 15 [JEE Advanced 2012]

$6 + \log_{\left(\frac{3}{2}\right)}\left(\frac{4}{9}\right) = 6 + \log_{\left(\frac{3}{2}\right)}\left(\frac{2}{3}\right)^2 = 6 + 2(-1) = 4$

Ans:4

Find the value of the expression:

$$6 + \log_{3/2}\left(\frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\cdots\infty}}}\right)$$

$$y = \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\cdots\infty}}}$$

$\Rightarrow y$

$$y = \frac{1}{3\sqrt{2}}\sqrt{4-y}$$

$$18y^2 = (4-y)$$

$$18y^2 + y - 4 = 0$$

-72

$+9 \quad -8$

$$18y^2 + 9y - 8y - 4 = 0$$

$$9y(2y+1) - 4(2y+1) = 0$$

$y = -\frac{1}{2}$ Reject $\qquad y = \frac{4}{9}$

Que $\sqrt{6+\sqrt{6+\sqrt{6+\cdots\infty}}}$

$$y=\sqrt{6+\underbrace{\sqrt{6+\sqrt{6+\cdots\infty}}}_{=y}}$$

$$y=\sqrt{6+y}$$

$$y^2=6+y$$

$$y^2-y-6=0$$

-6: -3, $+2$

$y=3$ // $y=-2$ Reject

Main-2021

$$y=3+\cfrac{1}{4+\underbrace{\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{3+\cdots\infty}}}}_{=y}}$$

$$y=3+\cfrac{1}{4+\cfrac{1}{y}}$$

$$y-3=\frac{y}{4y+1}$$

$$4y^2+\cancel{y}-12y-3-\cancel{y}=0$$

$$4y^2-12y-3=0$$

Ans: $\frac{3}{2}+\sqrt{3}$ //

Example: 16 [JEE Advanced 2018] Ans:8

Find the value of the expression:

$$\left((\log_2 9)^2\right)^{\frac{1}{\log_2(\log_2 9)}} \times (\sqrt{7})^{\frac{1}{\log_4 7}}$$

$4 \times 2 = 8$

$$\left((\log_2 9)^2\right)^{\frac{1}{\log_2(\log_2 9)}} \times \left(7^{\frac{1}{2}}\right)^{\log_7 4}$$

$$\left(a^2\right)^{\frac{1}{\log_2 a}}$$

$$a^{2 \cdot \log_a 2} = a^{\log_a 2^2} = 4$$

$$7^{\frac{1}{2} \cdot \log_7 4}$$

$$7^{\log_7 (4)^{1/2}}$$

$$= 2$$

Example: 17 [JEE Advanced 2024] Ans:8

Let $a = 3\sqrt{2}$ and $b = \frac{1}{5^{1/6}\sqrt{6}}$. If $x, y \in \mathbb{R}$ are such that

$$3x + 2y = \log_a(18)^{\frac{5}{4}} \quad \text{and} \quad 2x - y = \log_b(\sqrt{1080}),$$

then $4x + 5y$ is equal to ______.

Sabko prime factor me tod do

$$\log_a (18)^{5/4} = \frac{5}{4}\log_{3\sqrt{2}}(18) = \frac{5}{4}\log_{\sqrt{18}} 18 = \frac{5}{4}\times 2 \cdot \log_{18} 18 = \frac{5}{2}$$

$$\log_b \sqrt{1080} = \log_{\left(5^{-\frac{1}{6}}\,6^{-\frac{1}{2}}\right)} \left(5^{1}\cdot 6^{3}\right)^{\frac{1}{2}} = -1 \log_{\left(5^{\frac{1}{6}}\cdot 6^{\frac{1}{2}}\right)} \left(5^{\frac{1}{2}}\cdot 6^{\frac{3}{2}}\right) = -3 \log_{\square}\square = -3$$

$$1080 = (108)\times 10$$
$$= 2\times 2\times 3\times 3 \times 3\times 2\times 5$$
$$= 2^3 \cdot 3^3 \cdot 5^1$$
$$1080 = (6)^3(5)^1$$

Result 12

$$a^{\log_c b} = b^{\log_c a}$$

Result 13

$$a^{\sqrt{\log_a b}} = b^{\sqrt{\log_b a}}$$

$$a^{\sqrt{\log_a b}} = b^{\sqrt{\log_b a}}$$

$$LHS = a^{\sqrt{\log_a b}} = a^{\sqrt{\log_a b \cdot \log_a b \cdot \log_b a}}$$

$$= a^{(\log_a b)\cdot\sqrt{\log_b a}}$$

$$= \left(a^{\log_a b}\right)^{\sqrt{\log_b a}}$$

$$= b^{\sqrt{\log_b a}}$$

Example: 18 [JEE Main 2022] Ans:(B)

If α, β are the roots of the equation

$$x^2 - (5 + \cancel{3^{\sqrt{\log_3 5}}} - \cancel{5^{\sqrt{\log_5 3}}})x + 3(\cancel{3^{(\log_3 5)^{\frac{1}{3}}}} - \cancel{5^{(\log_5 3)^{\frac{2}{3}}}} - 1) = 0$$

then the equation, whose roots are $\alpha + \frac{1}{\beta}$ and $\beta + \frac{1}{\alpha}$, is:

(A) $3x^2 - 20x - 12 = 0$

(B) $3x^2 - 10x - 4 = 0$

(C) $3x^2 - 10x + 2 = 0$

(D) $3x^2 - 20x + 16 = 0$

$$x^2 - (5)x + 3(-1) = 0$$

$$x^2 - 5x - 3 = 0 \quad \alpha, \beta$$

Basic Mathematics: Section 8 (Part-03)

Logarithmic Inequality

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Graphs of Logarithmic Functions

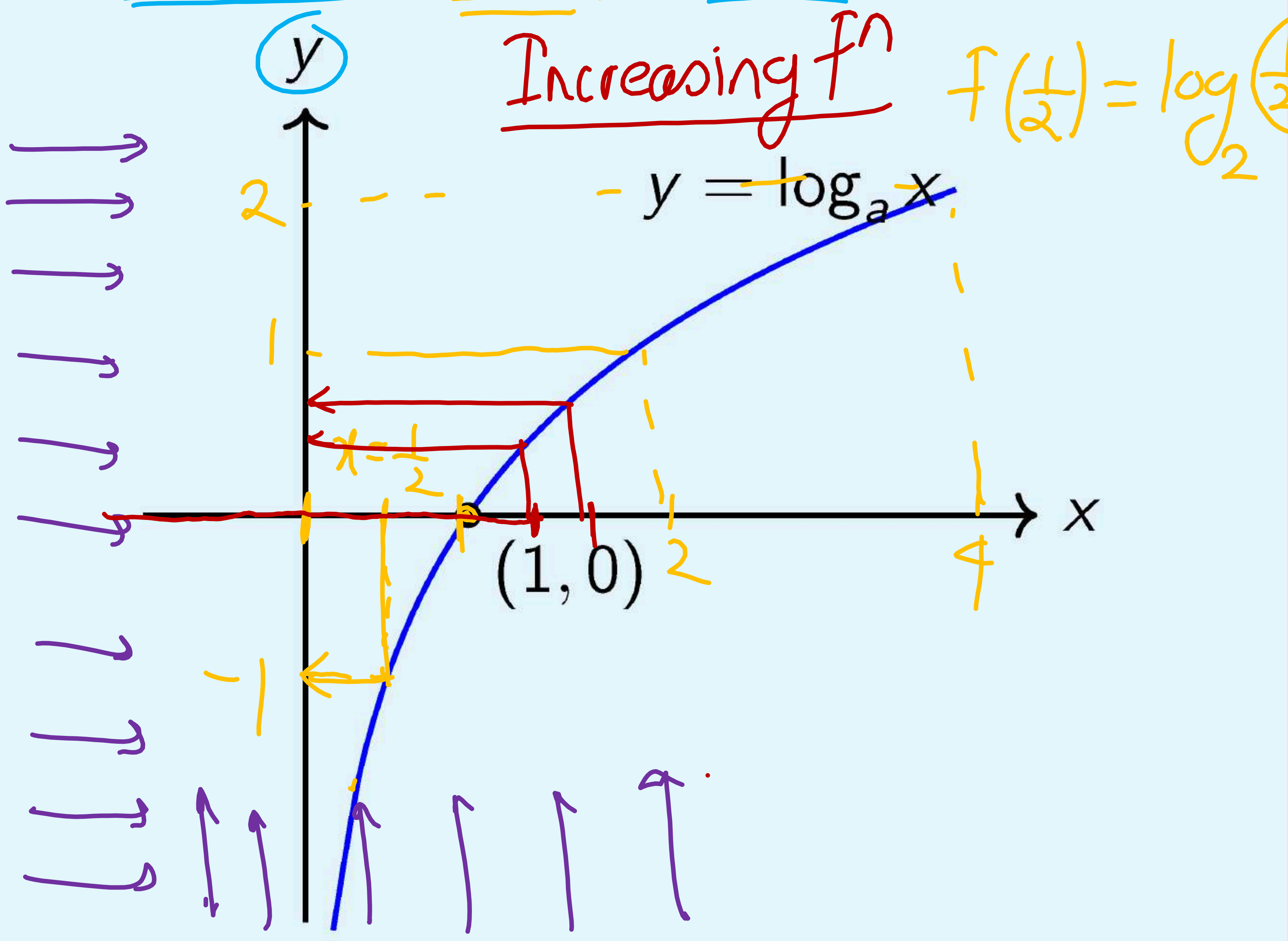
$f(x) = \log_a x$

Case 1: Base $a > 1$

Example: $y = \log_2 x$ $\quad f(x) = \log_2 x$

- **Domain:** $(0, \infty)$; **Range:** $(-\infty, \infty)$

Increasing f^n $\quad f(\tfrac{1}{2}) = \log_2 (\tfrac{1}{2}) = -1$

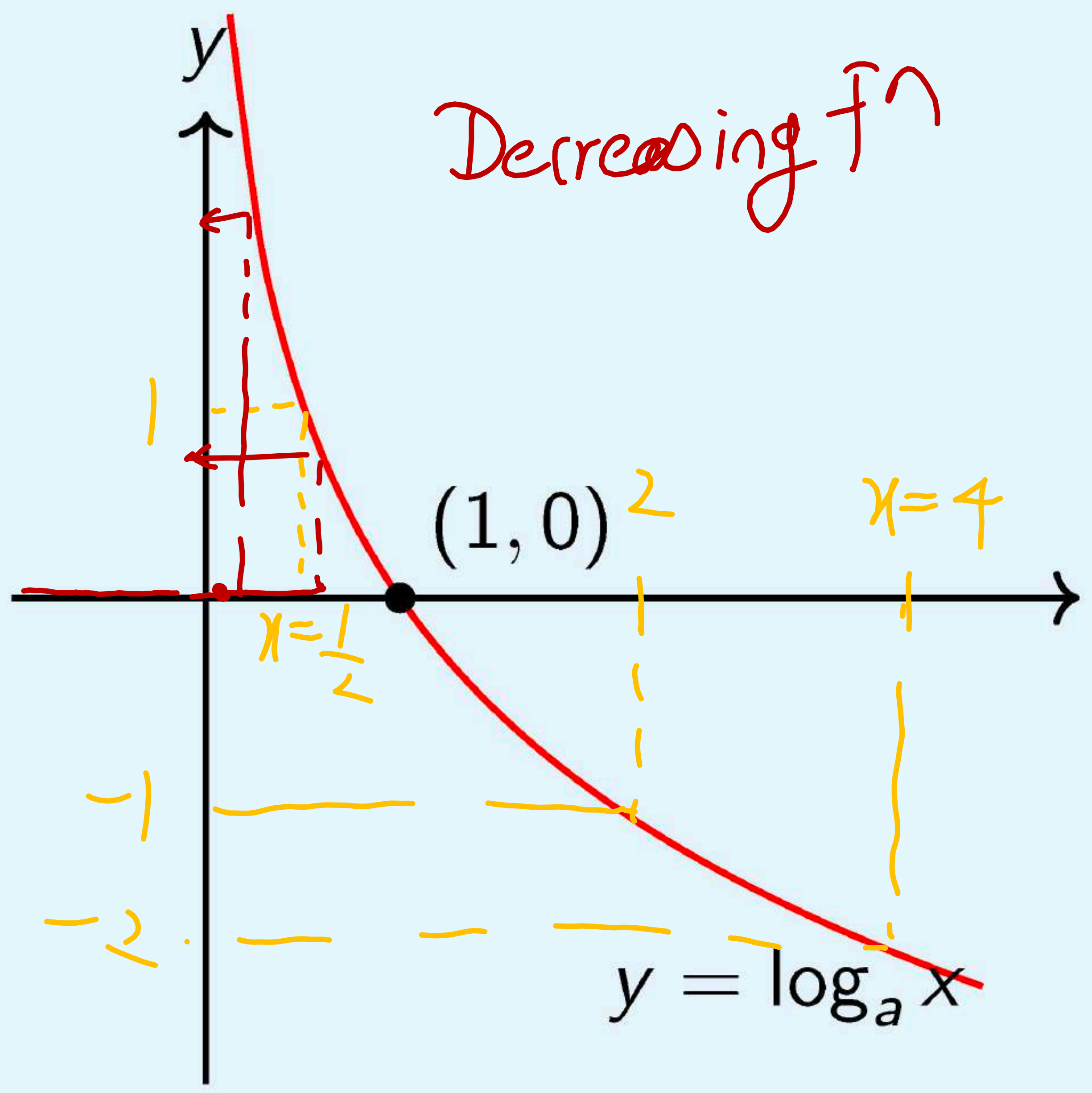


Case 2: $0 < a < 1$

Example: $y = \log_{1/2} x = -\log_2 x$

- **Domain:** $(0, \infty)$; **Range:** $(-\infty, \infty)$

Decreasing f^n

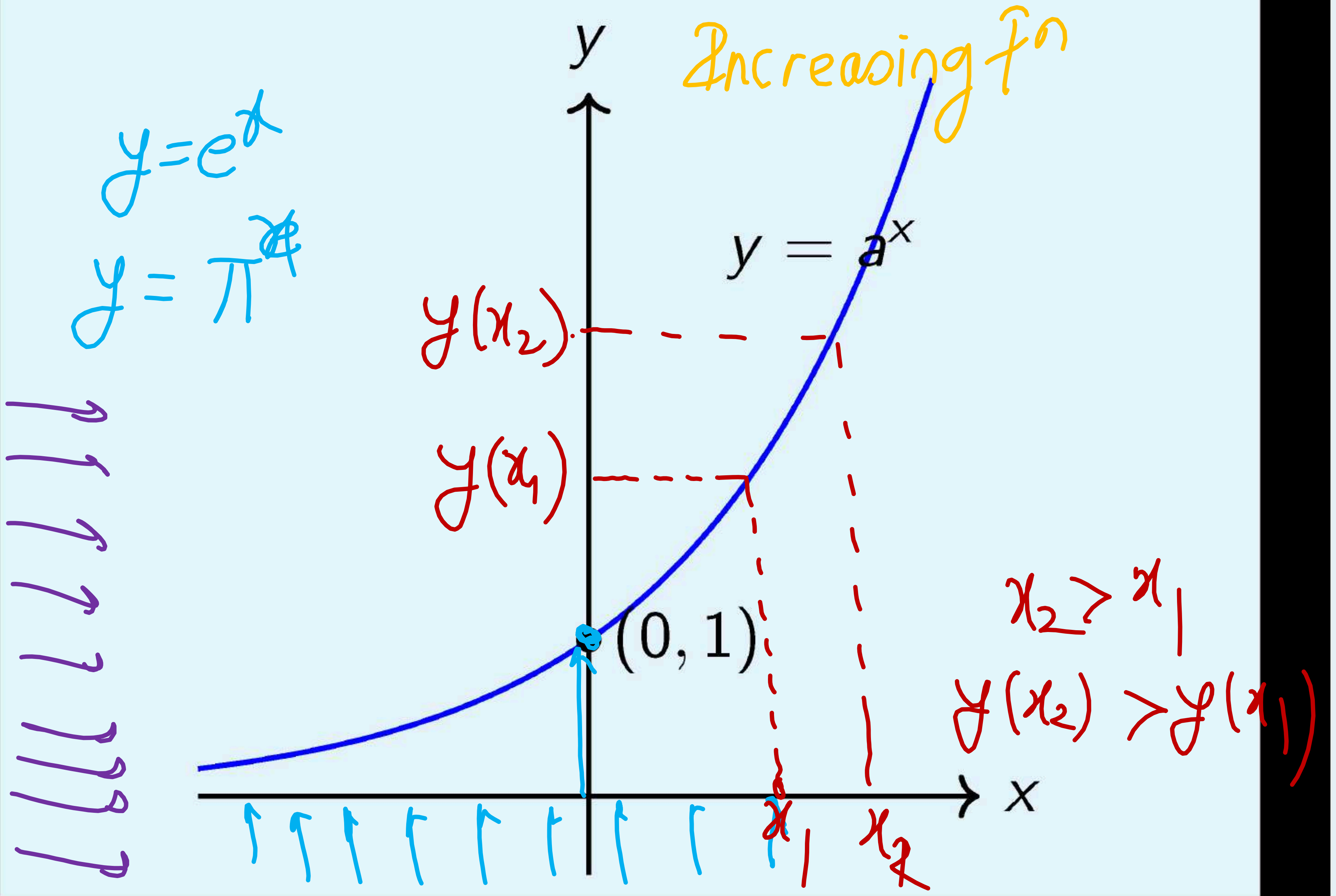


Graphs of Exponential Functions $f(x) = a^x$

Case 1: Base $a > 1$

Example: $y = 2^x$

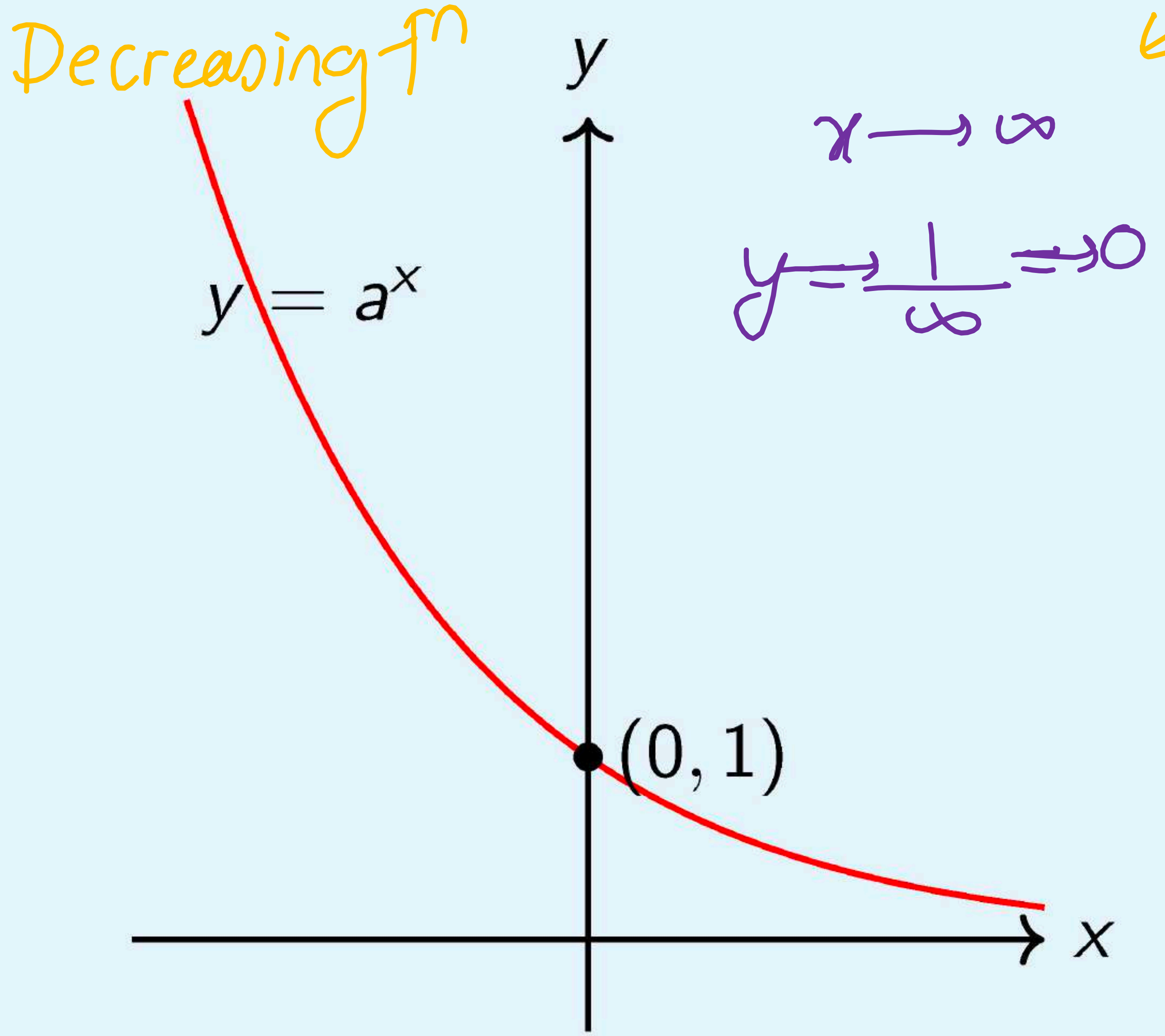
- **Domain:** $(-\infty, \infty)$; **Range:** $(0, \infty)$



Case 2: $\boxed{0 < a < 1}$

Example: $y = (1/2)^x = \frac{1}{2^x}$ $y = e^{-x} = \frac{1}{e^x}$

- **Domain:** $(-\infty, \infty)$; **Range:** $(0, \infty)$



Logarithmic Inequalities: The Role of the Base

$\cancel{\log_2} x_1 > \cancel{\log_2} x_2 \Rightarrow x_1 > x_2$ $\cancel{\log_{\frac{1}{5}}} x_1 > \cancel{\log_{\frac{1}{5}}} x_2 \Rightarrow x_1 < x_2$

Case 1: Base $a > 1$

The function is increasing so,

$$\log_a x > \log_a y \iff x > y$$

Case 2: $0 < a < 1$

The function is decreasing so,

$$\log_a x > \log_a y \iff x < y$$

Example

$x_1 > x_2$ $\log_3 x_1 > \log_3 x_2$

$$81 > 27 \iff \log_3 81 > \log_3 27$$

$4 > 3$

Example

$$81 > 27 \iff \log_{1/3} 81 < \log_{1/3} 27$$

$-\log_3 81$ $-\log_3 27$

$-4 < -3$

Example: 1

Solve the inequality:

$$\log_{1/3}(5x-1) > 0$$

<u>Step-1</u> Find domain

$5x-1>0$

$5x>1$

$x>\frac{1}{5}$ // — 1

<u>Step-2</u>

$\cancel{\log}_{\frac{1}{3}}(5x-1)>0$

$(5x-1)<\left(\frac{1}{3}\right)^0$

$5x-1<1$

$5x<2$

$x<\frac{2}{5}$ — 2

<u>Step-3</u> Intersection with domain

1 $\cap$ 2

$x\in\left(\frac{1}{5},\frac{2}{5}\right)$

Example: 2

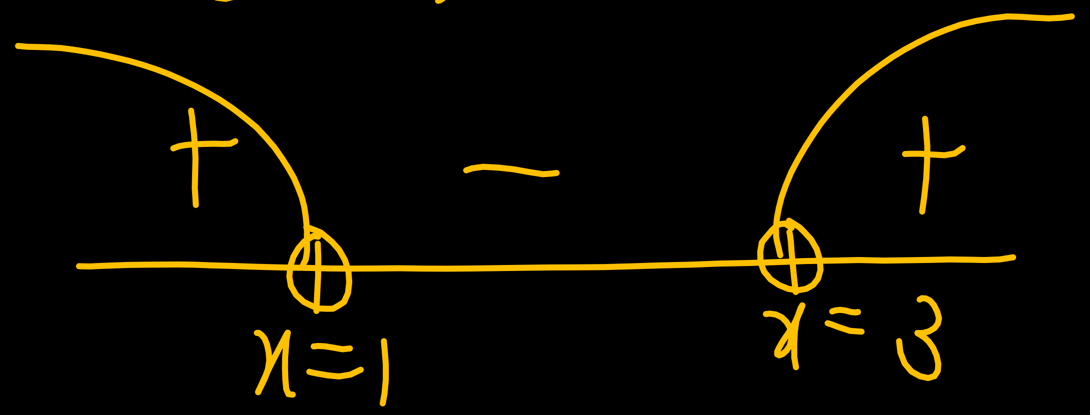
Solve the inequality:

$$\log_8(x^2 - 4x + 3) \leq 1$$

Step-1 Domain

$$x^2 - 4x + 3 > 0$$

$$(x-3)(x-1) > 0$$



Step-2

$$\log_8(x^2 - 4x + 3) \leq 1$$

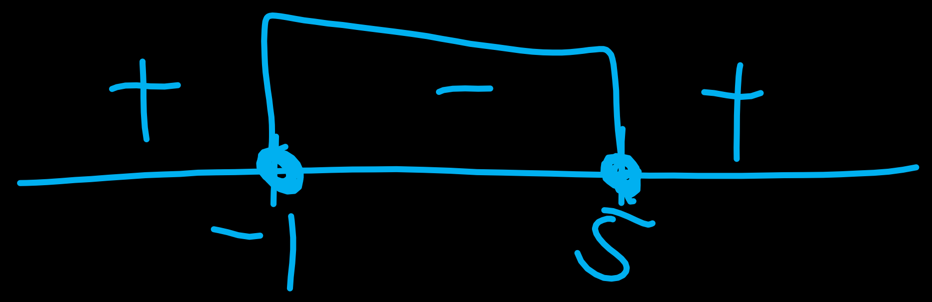
$$x^2 - 4x + 3 \leq 8^1$$

$$x^2 - 4x - 5 \leq 0$$

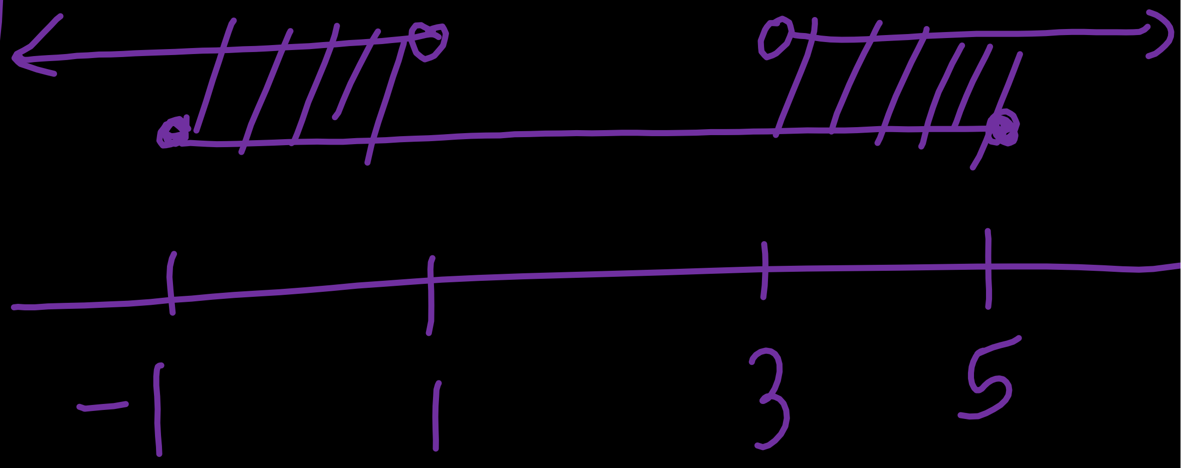
−5

−5 +1

$$(x-5)(x+1) \leq 0$$



Step-3 Intersection with domain



Ans: $[-1, 1) \cup (3, 5]$

Example: 3

Solve the inequality:

$$\log_{1/3}(\log_4(x^2-5)) > 0$$

Domain:

C-1 $\log_4(x^2-5) > 0$

$\boxed{x^2-5} > 1$

$(x-\sqrt{6})(x+\sqrt{6}) > 0$

$+ \quad -\sqrt{6} \quad - \quad -\sqrt{5} \quad \sqrt{5} \quad \sqrt{6} \quad +$ — ①

C-2 $\boxed{x^2-5} > 0$

$+ \quad -\sqrt{5} \quad - \quad \sqrt{5} \quad +$

$\log_{\frac{1}{3}}\left(\log_4(x^2-5)\right) > 0$

$\log_4(x^2-5) < \left(\frac{1}{3}\right)^0$

$x^2-5 < 4^1$

$x^2-9 < 0$

$(x-3)(x+3) < 0$ — ②

$-3 \quad 3$

① ∩ ②

$-3 \quad -\sqrt{6} \quad \sqrt{6} \quad 3$

$x \in (-3, -\sqrt{6}) \cup (\sqrt{6}, 3)$

Example: 4 [JEE Main 2025] — Ans:(A)

① ∩ ② $2 > x^2+4x+5 > 0$

Let the domains of the functions $f(x) = \log_4 \log_3 \log_7(8 - \log_2(x^2+4x+5))$ and $g(x) = \sin^{-1}\left(\frac{7x+10}{x-2}\right)$ be (α, β) and $[\gamma, \delta]$, respectively. Then $\alpha^2+\beta^2+\gamma^2+\delta^2$ is equal to :

$(-3)^2 + (-1)^2 + (-2)^2 + (-1)^2 = 15$

(A) 15 (B) 13 (C) 16 (D) 14

$f(x) = \log_4 \log_3 \log_7 \left(8 - \log_2(x^2+4x+5)\right)$

C-2 $\log_7\left(8 - \log_2(x^2+4x+5)\right) > 0$

C-1 $\log_3\left(\log_7\left(8 - \log_2(x^2+4x+5)\right)\right) > 0$

$\log_7\left(8-\log_2(x^2+4x+5)\right) > 1$

$8 - \log_2(x^2+4x+5) > 7$

$1 > \log_2(x^2+4x+5)$ — ①

C-3 $8 - \log_2(x^2+4x+5) > 0$

C-4 $x^2+4x+5 > 0$ — ②

Example: 5 (Variable Base)

Solve the inequality:

$$\log_x(x^3 - x^2 - 2x) < 3$$

Step 1 Domain

$x^3 - x^2 - 2x > 0$

$x(x^2 - x - 2) > 0$

$x(x-2)(x+1) > 0$

$-1 \quad 0 \quad 1 \quad 2$

Final Ans: $(2, \infty)$

Base $= x$

Case I: Base > 1 Ans: $x \in (1, \infty)$ ①

$x > 1$

$\cancel{a} \cdot b > \cancel{a} \cdot c$

$a = 0$

$\frac{\cancel{a} \cdot b}{c \cdot \cancel{a}}$ $a \neq 0$

$\cancel{\log_x}(x^3 - x^2 - 2x) < 3$

$\cancel{x^3} - x^2 - 2x < \cancel{x^3}$

$-(x^2 + 2x) < 0$

$x(x+2) > 0$

$+ \quad - \quad +$

$-2 \quad 0$

Case II: $0 <$ Base < 1 Ans: $x \in \phi$

$0 < x < 1$

$\cancel{\log_x}(x^3 - x^2 - 2x) < 3$

$\cancel{x^3} - x^2 - 2x > \cancel{x^3}$

$-(x^2 + 2x) > 0$

$x(x+2) < 0$

$+ \quad - \quad +$

$-2 \quad 0$

Example: 6 [JEE Main 2023] (Variable Base) Ans:(C)

The number of integral solutions x of $\log_{\left(x+\frac{7}{2}\right)}\left(\frac{x-7}{2x-3}\right)^2 \geq 0$ is:

(A) 7 (B) 8 (C) 6 (D) 5

Case I : Base > 1

$x+\frac{7}{2} > 1$

$x > 1-\frac{7}{2}$

$x > -\frac{5}{2}$

Case II : 0 < Base < 1

$0 < x+\frac{7}{2} < 1$

$-\frac{7}{2} < x < 1-\frac{7}{2}$

$-\frac{7}{2} < x < -\frac{5}{2}$

Irrational Equation & Inequality

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Example: 1

Solve the irrational inequality:

$$\underbrace{4 - x}_{} > \underbrace{\sqrt{3x - x^2}}_{\text{always +ve}}$$

$f(x) > g(x)$

$4 \geq x$

Case I: $\underline{4-x} \geq 0$

$\underbrace{4-x}_{+ve} > \underbrace{\sqrt{3x-x^2}}_{+ve}$

Square

$(4-x)^2 > (3x-x^2)$

Case II: $4-x < 0$

$x > 4$

$\underbrace{4-x}_{-ve} > \underbrace{\sqrt{3x-x^2}}_{+ve}$

No sol^n $x \in \phi$

Domain:

$3x - x^2 \geq 0$

Example: 2

Solve the irrational inequality:

$$\underbrace{\sqrt{x^2+4x-5}}_{\text{always +ve}} > \underline{x-3}$$

Case I $x-3 \geq 0$

$x \geq 3$

$$\underbrace{\sqrt{x^2+4x-5}}_{+ve} > \underbrace{x-3}_{+ve}$$

Square

$$x^2+4x-5 > (x-3)^2$$

Case II $x-3 < 0$

$x < 3$

$$\underbrace{\sqrt{x^2+4x-5}}_{+ve} > \underbrace{x-3}_{-ve}$$

Always

$x \in R$

Domain

Solution: Example 2

The inequality is $\sqrt{x^2 + 4x - 5} > x - 3$.

Step 1: Find the Domain

$$x^2 + 4x - 5 \geq 0$$
$$(x+5)(x-1) \geq 0$$

This gives the domain $x \in (-\infty, -5] \cup [1, \infty)$. (Equation 1)

Step 2: Consider cases for $(x - 3)$

Case I: $x - 3 \geq 0 \implies x \geq 3$

Both sides are non-negative. We can square both sides:

$$\begin{aligned} x^2 + 4x - 5 &> (x-3)^2 \\ x^2 + 4x - 5 &> x^2 - 6x + 9 \\ 10x &> 14 \\ x &> 1.4 \end{aligned}$$

The solution for this case is the intersection of $x \geq 3$ and $x > 1.4$, which is $x \in [3, \infty)$. (Equation 2)

Case II: $x - 3 < 0 \implies x < 3$

The LHS is non-negative and the RHS is negative.

The inequality is always true.

The solution is all x such that $x < 3$. (Equation 3)

Step 3: Combine Solutions

Union of Case I and Case II:

Intersection with Domain:

The final solution is:

$$(-\infty, -5] \cup [1, \infty)$$

Example: 3

Ans: $x \in (4, 6]$

Solve the irrational inequality:

$$\underbrace{\sqrt{3x-10}}_{+ve} > \underbrace{\sqrt{6-x}}_{+ve}$$

Square

+

Intersection with domain

$3x-10 \geq 0$ and $6-x \geq 0$

Example: 4

Solve the irrational equation:

$$\sqrt{2x-3}+\sqrt{4x+1}=4$$

Square

$$(2x-3)+(4x+1)+2\sqrt{2x-3}\sqrt{4x+1}=16$$

$$\left(2\sqrt{2x-3}\sqrt{4x+1}\right)=\left(16-6x+2\right)$$

Again square

Solution: Example 4

The equation is $\sqrt{2x-3}+\sqrt{4x+1}=4$.

Step 1: Find the Domain

$$2x-3 \geq 0 \implies x \geq \tfrac{3}{2}$$

$$4x+1 \geq 0 \implies x \geq -\tfrac{1}{4}$$

The intersection gives the domain: $x \in [\frac{3}{2}, \infty)$.

Step 2: Solve the Equation

Isolate one of the square roots:

$$\sqrt{4x+1} = 4 - \sqrt{2x-3}$$

Square both sides:

$$4x+1 = (4-\sqrt{2x-3})^2$$
$$4x+1 = 16 - 8\sqrt{2x-3} + (2x-3)$$
$$x-6 = -4\sqrt{2x-3}$$

Square both sides again:

$$(x-6)^2 = (-4\sqrt{2x-3})^2$$
$$x^2 - 12x + 36 = 16(2x-3)$$
$$x^2 - 44x + 84 = 0$$
$$(x-2)(x-42) = 0$$

The potential solutions are $x = 2$ and $x = 42$.

Step 3: Check Solutions with Domain or Directly Put and Check

The domain is $[\frac{3}{2}, \infty)$. Both $x = 2$ and $x = 42$ are in the domain.

$x = 2$ satisfies this condition.

$x = 42$ does **not** satisfy this condition.

The only valid solution is $x = 2$.