HI EVERYONE,

THE REAL LEARNING IN MATHEMATICS HAPPENS WHEN YOU ACTIVELY ENGAGE WITH A PROBLEM, EXPLORE DIFFERENT METHODS, AND WORK THROUGH CHALLENGES. THEREFORE, WE STRONGLY ENCOURAGE YOU TO USE THIS SOLUTION KEY RESPONSIBLY.

PLEASE ATTEMPT ALL THE PROBLEMS ON YOUR OWN FIRST, GIVING THEM YOUR BEST AND MOST HONEST EFFORT. THESE SOLUTIONS ARE TO HELP YOU GET UNSTUCK ON A PROBLEM AFTER YOU HAVE ALREADY TRIED YOUR BEST.

YOUR EFFORT AND DEDICATION ARE THE TRUE KEYS TO SUCCESS.

Trigonometric Ratios & Identities

Topic: SL LONEY Exercises

Sub: Mathematics | **Assignment: 03 Solutions** | Prof. Chetan Sir

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Index of Questions

Que.1: Solve the following (SL.LONEY Ex.13)

1. If $\sin\alpha = \frac{45}{53}$ and $\sin\beta = \frac{33}{65}$, find the values of $\sin(\alpha-\beta)$ and $\sin(\alpha+\beta)$.

Solution:

First, we find the values of $\cos\alpha$ and $\cos\beta$ using the identity $\cos^2\theta = 1-\sin^2\theta$. We assume α and β are acute angles.

$$\cos\alpha = \sqrt{1-\left(\frac{45}{53}\right)^2} = \sqrt{\frac{53^2-45^2}{53^2}} = \sqrt{\frac{2809-2025}{2809}} = \sqrt{\frac{784}{2809}} = \frac{28}{53}.$$

$$\cos\beta = \sqrt{1-\left(\frac{33}{65}\right)^2} = \sqrt{\frac{65^2-33^2}{65^2}} = \sqrt{\frac{4225-1089}{4225}} = \sqrt{\frac{3136}{4225}} = \frac{56}{65}.$$

Now, we use the sum and difference formulas for sine.

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ &= \left(\frac{45}{53}\right)\left(\frac{56}{65}\right) + \left(\frac{28}{53}\right)\left(\frac{33}{65}\right) \\ &= \frac{2520}{3445} + \frac{924}{3445} = \mathbf{\frac{3444}{3445}}.\end{aligned}$$

$$
\begin{aligned}
\sin(\alpha-\beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \\
&= \left(\frac{45}{53}\right)\left(\frac{56}{65}\right) - \left(\frac{28}{53}\right)\left(\frac{33}{65}\right) \\
&= \frac{2520}{3445} - \frac{924}{3445} = \mathbf{\frac{1596}{3445}}.
\end{aligned}
$$

2. If $\sin\alpha = \frac{15}{17}$ and $\cos\beta = \frac{12}{13}$, find the values of $\sin(\alpha+\beta)$, $\cos(\alpha-\beta)$ and $\tan(\alpha+\beta)$.

Solution:

First, find $\cos\alpha$ and $\sin\beta$, assuming acute angles.

$$
\begin{aligned}
\cos\alpha &= \sqrt{1-\sin^2\alpha} = \sqrt{1-\left(\frac{15}{17}\right)^2} = \sqrt{\frac{289-225}{289}} = \sqrt{\frac{64}{289}} = \frac{8}{17}. \\
\sin\beta &= \sqrt{1-\cos^2\beta} = \sqrt{1-\left(\frac{12}{13}\right)^2} = \sqrt{\frac{169-144}{169}} = \sqrt{\frac{25}{169}} = \frac{5}{13}.
\end{aligned}
$$

Now, we find the required values.

$$
\begin{aligned}
\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\
&= \left(\frac{15}{17}\right)\left(\frac{12}{13}\right) + \left(\frac{8}{17}\right)\left(\frac{5}{13}\right) = \frac{180+40}{221} = \mathbf{\frac{220}{221}}.
\end{aligned}
$$

$$
\begin{aligned}
\cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\
&= \left(\frac{8}{17}\right)\left(\frac{12}{13}\right) + \left(\frac{15}{17}\right)\left(\frac{5}{13}\right) = \frac{96+75}{221} = \mathbf{\frac{171}{221}}.
\end{aligned}
$$

For $\tan(\alpha+\beta)$, we find $\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{15/17}{8/17} = \frac{15}{8}$ and $\tan\beta = \frac{\sin\beta}{\cos\beta} = \frac{5/13}{12/13} = \frac{5}{12}$.

$$
\begin{aligned}
\tan(\alpha+\beta) &= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} = \frac{\frac{15}{8}+\frac{5}{12}}{1-\left(\frac{15}{8}\right)\left(\frac{5}{12}\right)} \\
&= \frac{\frac{45+10}{24}}{1-\frac{75}{96}} = \frac{55/24}{\frac{96-75}{96}} = \frac{55/24}{21/96} = \frac{55}{24}\times\frac{96}{21} = \frac{55\times 4}{21} = \mathbf{\frac{220}{21}}.
\end{aligned}
$$

3. Prove that: $\cos(45^\circ - A)\cos(45^\circ - B) - \sin(45^\circ - A)\sin(45^\circ - B) = \sin(A+B)$.

Solution:

The Left Hand Side (LHS) matches the identity $\cos(X+Y) = \cos X\cos Y - \sin X\sin Y$. Let $X = 45^\circ - A$ and $Y = 45^\circ - B$.

$$
\begin{aligned}
\text{LHS} &= \cos(X+Y) = \cos((45^\circ - A) + (45^\circ - B)) \\
&= \cos(90^\circ - (A+B)) \\
&= \sin(A+B) \qquad \text{[Using the co-function identity } \cos(90^\circ-\theta) = \sin\theta\text{]}
\end{aligned}
$$

LHS = RHS. Hence, proved.

4. Prove that: $\sin(45^\circ + A)\cos(45^\circ - B) + \cos(45^\circ + A)\sin(45^\circ - B) = \cos(A-B)$.

Solution:

The LHS matches the identity $\sin(X+Y) = \sin X \cos Y + \cos X \sin Y$. Let $X = 45^\circ + A$ and $Y = 45^\circ - B$.

$$\begin{aligned} \text{LHS} &= \sin(X+Y) = \sin((45^\circ + A) + (45^\circ - B)) \\ &= \sin(90^\circ + (A - B)) \\ &= \cos(A - B) && \text{[Using the co-function identity } \sin(90^\circ + \theta) = \cos\theta] \end{aligned}$$

LHS = RHS. Hence, proved.

5. Prove that: $\frac{\sin(A-B)}{\cos A \cos B} + \frac{\sin(B-C)}{\cos B \cos C} + \frac{\sin(C-A)}{\cos C \cos A} = 0$.

Solution:

We expand each term on the LHS using the difference identity for sine, $\sin(X - Y) = \sin X \cos Y - \cos X \sin Y$.

$$\begin{aligned} \text{Term 1} &= \frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B} = \frac{\sin A \cos B}{\cos A \cos B} - \frac{\cos A \sin B}{\cos A \cos B} = \tan A - \tan B. \\ \text{Term 2} &= \frac{\sin B \cos C - \cos B \sin C}{\cos B \cos C} = \tan B - \tan C. \\ \text{Term 3} &= \frac{\sin C \cos A - \cos C \sin A}{\cos C \cos A} = \tan C - \tan A. \end{aligned}$$

Now, we add the simplified terms.

$$\text{LHS} = (\tan A - \tan B) + (\tan B - \tan C) + (\tan C - \tan A) = 0.$$

LHS = RHS. Hence, proved.

6. Prove that: $\sin 105^\circ + \cos 105^\circ = \cos 45^\circ$.

Solution:

We evaluate the LHS using the sum formulas for sine and cosine.

$$\begin{aligned} \text{LHS} &= \sin(60^\circ + 45^\circ) + \cos(60^\circ + 45^\circ) \\ &= (\sin 60^\circ \cos 45^\circ + \cos 60^\circ \sin 45^\circ) + (\cos 60^\circ \cos 45^\circ - \sin 60^\circ \sin 45^\circ) \\ &= \left(\frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} + \frac{1}{2} \cdot \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{2} \cdot \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}}\right) \\ &= \frac{\sqrt{3}+1}{2\sqrt{2}} + \frac{1-\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{3}+1+1-\sqrt{3}}{2\sqrt{2}} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}. \end{aligned}$$

The Right Hand Side (RHS) is $\cos 45^\circ = \frac{1}{\sqrt{2}}$. LHS = RHS. Hence, proved.

7. Prove that: $\sin 75^\circ - \sin 15^\circ = \cos 105^\circ + \cos 15^\circ$.

Solution:

We evaluate both sides using sum-to-product formulas. For the LHS:

$$\begin{aligned} \text{LHS} = \sin 75^\circ - \sin 15^\circ &= 2\cos\left(\frac{75+15}{2}\right)\sin\left(\frac{75-15}{2}\right) \\ &= 2\cos 45^\circ \sin 30^\circ = 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) = \frac{1}{\sqrt{2}}. \end{aligned}$$

For the RHS:

$$\begin{aligned} \text{RHS} = \cos 105^\circ + \cos 15^\circ &= 2\cos\left(\frac{105+15}{2}\right)\cos\left(\frac{105-15}{2}\right) \\ &= 2\cos 60^\circ \cos 45^\circ = 2\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}. \end{aligned}$$

LHS = RHS. Hence, proved.

8. Prove that: $\cos\alpha\cos(\gamma-\alpha)-\sin\alpha\sin(\gamma-\alpha)=\cos\gamma$**.**

Solution:
The LHS matches the identity $\cos(X+Y)=\cos X\cos Y-\sin X\sin Y$. Let $X=\alpha$ and $Y=\gamma-\alpha$.

$$\text{LHS}=\cos(\alpha+(\gamma-\alpha))=\cos(\alpha+\gamma-\alpha)=\cos(\gamma).$$

LHS = RHS. Hence, proved.

9. Prove that: $\cos(\alpha+\beta)\cos\gamma-\cos(\beta+\gamma)\cos\alpha=\sin\beta\sin(\gamma-\alpha)$**.**

Solution:
We expand the terms on the LHS.

$$\begin{aligned}\text{LHS}&=(\cos\alpha\cos\beta-\sin\alpha\sin\beta)\cos\gamma-(\cos\beta\cos\gamma-\sin\beta\sin\gamma)\cos\alpha\\&=(\cos\alpha\cos\beta\cos\gamma-\sin\alpha\sin\beta\cos\gamma)-(\cos\alpha\cos\beta\cos\gamma-\sin\beta\sin\gamma\cos\alpha)\\&=\cos\alpha\cos\beta\cos\gamma-\sin\alpha\sin\beta\cos\gamma-\cos\alpha\cos\beta\cos\gamma+\sin\beta\sin\gamma\cos\alpha\\&=\sin\beta\sin\gamma\cos\alpha-\sin\alpha\sin\beta\cos\gamma\\&=\sin\beta(\sin\gamma\cos\alpha-\cos\gamma\sin\alpha)\\&=\sin\beta\sin(\gamma-\alpha).\end{aligned}$$

LHS = RHS. Hence, proved.

10. Prove that: $\sin(n+1)A\sin(n-1)A+\cos(n+1)A\cos(n-1)A=\cos 2A$**.**

Solution:
The LHS matches the identity $\cos(X-Y)=\cos X\cos Y+\sin X\sin Y$. Let $X=(n+1)A$ and $Y=(n-1)A$.

$$\begin{aligned}\text{LHS}&=\cos((n+1)A-(n-1)A)\\&=\cos(nA+A-nA+A)=\cos(2A).\end{aligned}$$

LHS = RHS. Hence, proved.

11. Prove that: $\sin(n+1)A\sin(n+2)A+\cos(n+1)A\cos(n+2)A=\cos A$**.**

Solution:
The LHS matches the identity $\cos(X-Y)=\cos X\cos Y+\sin X\sin Y$. Let $X=(n+2)A$ and $Y=(n+1)A$.

$$\begin{aligned}\text{LHS}&=\cos((n+2)A-(n+1)A)\\&=\cos(nA+2A-nA-A)=\cos(A).\end{aligned}$$

LHS = RHS. Hence, proved.

Que.2: Prove that (SL.LONEY Ex.14)

1. $\frac{\sin 7\theta - \sin 5\theta}{\cos 7\theta + \cos 5\theta} = \tan\theta$

Solution:

We use the sum-to-product formulas on the LHS.

$$\begin{aligned}\text{LHS} &= \frac{2\cos\left(\frac{7\theta+5\theta}{2}\right)\sin\left(\frac{7\theta-5\theta}{2}\right)}{2\cos\left(\frac{7\theta+5\theta}{2}\right)\cos\left(\frac{7\theta-5\theta}{2}\right)} \\ &= \frac{2\cos(6\theta)\sin(\theta)}{2\cos(6\theta)\cos(\theta)} = \frac{\sin\theta}{\cos\theta} = \tan\theta.\end{aligned}$$

LHS = RHS. Hence, proved.

2. $\frac{\cos 6\theta - \cos 4\theta}{\sin 6\theta + \sin 4\theta} = -\tan\theta$

Solution:

Using sum-to-product formulas on the LHS:

$$\begin{aligned}\text{LHS} &= \frac{-2\sin\left(\frac{6\theta+4\theta}{2}\right)\sin\left(\frac{6\theta-4\theta}{2}\right)}{2\sin\left(\frac{6\theta+4\theta}{2}\right)\cos\left(\frac{6\theta-4\theta}{2}\right)} \\ &= \frac{-2\sin(5\theta)\sin(\theta)}{2\sin(5\theta)\cos(\theta)} = -\frac{\sin\theta}{\cos\theta} = -\tan\theta.\end{aligned}$$

LHS = RHS. Hence, proved.

3. $\frac{\sin 7A - \sin A}{\sin 8A - \sin 2A} = \cos 4A \sec 5A$

Solution:

Using sum-to-product formulas on the LHS:

$$\begin{aligned}\text{LHS} &= \frac{2\cos\left(\frac{7A+A}{2}\right)\sin\left(\frac{7A-A}{2}\right)}{2\cos\left(\frac{8A+2A}{2}\right)\sin\left(\frac{8A-2A}{2}\right)} \\ &= \frac{2\cos(4A)\sin(3A)}{2\cos(5A)\sin(3A)} = \frac{\cos(4A)}{\cos(5A)} = \cos 4A \sec 5A.\end{aligned}$$

LHS = RHS. Hence, proved.

4. $\frac{\cos 2B + \cos 2A}{\cos 2B - \cos 2A} = \cot(A+B)\cot(A-B)$

Solution:

Using sum-to-product formulas on the LHS:

$$\begin{aligned}\text{LHS} &= \frac{2\cos\left(\frac{2B+2A}{2}\right)\cos\left(\frac{2B-2A}{2}\right)}{-2\sin\left(\frac{2B+2A}{2}\right)\sin\left(\frac{2B-2A}{2}\right)} \\ &= \frac{2\cos(A+B)\cos(B-A)}{-2\sin(A+B)\sin(B-A)} \\ &= \frac{\cos(A+B)\cos(A-B)}{-\sin(A+B)[-\sin(A-B)]} \qquad \text{[Using } \cos(-x) = \cos x \text{ and } \sin(-x) = -\sin x\text{]} \\ &= \frac{\cos(A+B)}{\sin(A+B)} \cdot \frac{\cos(A-B)}{\sin(A-B)} = \cot(A+B)\cot(A-B).\end{aligned}$$

LHS = RHS. Hence, proved.

5. $\frac{\sin 2A+\sin 2B}{\sin 2A-\sin 2B} = \frac{\tan(A+B)}{\tan(A-B)}$

Solution:

Using sum-to-product formulas on the LHS:

$$\begin{aligned}
\text{LHS} &= \frac{2\sin\left(\frac{2A+2B}{2}\right)\cos\left(\frac{2A-2B}{2}\right)}{2\cos\left(\frac{2A+2B}{2}\right)\sin\left(\frac{2A-2B}{2}\right)} \\
&= \frac{\sin(A+B)\cos(A-B)}{\cos(A+B)\sin(A-B)} \\
&= \frac{\sin(A+B)}{\cos(A+B)} \cdot \frac{\cos(A-B)}{\sin(A-B)} = \tan(A+B)\cot(A-B) = \frac{\tan(A+B)}{\tan(A-B)}.
\end{aligned}$$

LHS = RHS. Hence, proved.

6. $\frac{\sin A+\sin 2A}{\cos A-\cos 2A} = \cot\frac{A}{2}$

Solution:

Using sum-to-product formulas on the LHS:

$$\begin{aligned}
\text{LHS} &= \frac{2\sin\left(\frac{A+2A}{2}\right)\cos\left(\frac{A-2A}{2}\right)}{-2\sin\left(\frac{A+2A}{2}\right)\sin\left(\frac{A-2A}{2}\right)} \\
&= \frac{2\sin(3A/2)\cos(-A/2)}{-2\sin(3A/2)\sin(-A/2)} = \frac{\cos(A/2)}{-(-\sin(A/2))} = \frac{\cos(A/2)}{\sin(A/2)} = \cot\frac{A}{2}.
\end{aligned}$$

LHS = RHS. Hence, proved.

7. $\frac{\cos 2B-\cos 2A}{\sin 2B+\sin 2A} = \tan(A-B)$

Solution:

Using sum-to-product formulas on the LHS:

$$\begin{aligned}
\text{LHS} &= \frac{-2\sin\left(\frac{2B+2A}{2}\right)\sin\left(\frac{2B-2A}{2}\right)}{2\sin\left(\frac{2B+2A}{2}\right)\cos\left(\frac{2B-2A}{2}\right)} \\
&= \frac{-\sin(B-A)}{\cos(B-A)} = -\tan(B-A) = \tan(-(B-A)) = \tan(A-B).
\end{aligned}$$

LHS = RHS. Hence, proved.

8. $\cos(A+B)+\sin(A-B) = 2\sin(45^\circ+A)\cos(45^\circ+B)$

Solution:

We will simplify the RHS using the product-to-sum formula $2\sin X\cos Y = \sin(X+Y)+\sin(X-Y)$.

$$\begin{aligned}
\text{RHS} &= 2\sin(45^\circ+A)\cos(45^\circ+B) \\
&= \sin((45^\circ+A)+(45^\circ+B)) + \sin((45^\circ+A)-(45^\circ+B)) \\
&= \sin(90^\circ+A+B) + \sin(A-B) \\
&= \cos(A+B) + \sin(A-B).
\end{aligned}$$

RHS = LHS. Hence, proved.

9. $\frac{\sin(4A-2B)+\sin(4B-2A)}{\cos(4A-2B)+\cos(4B-2A)} = \tan(A+B)$

Solution:
Using sum-to-product formulas on the LHS:

$$\text{LHS} = \frac{2\sin\left(\frac{(4A-2B)+(4B-2A)}{2}\right)\cos\left(\frac{(4A-2B)-(4B-2A)}{2}\right)}{2\cos\left(\frac{(4A-2B)+(4B-2A)}{2}\right)\cos\left(\frac{(4A-2B)-(4B-2A)}{2}\right)}$$
$$= \frac{2\sin\left(\frac{2A+2B}{2}\right)\cos\left(\frac{6A-6B}{2}\right)}{2\cos\left(\frac{2A+2B}{2}\right)\cos\left(\frac{6A-6B}{2}\right)}$$
$$= \frac{\sin(A+B)\cos(3A-3B)}{\cos(A+B)\cos(3A-3B)} = \frac{\sin(A+B)}{\cos(A+B)} = \tan(A+B).$$

LHS = RHS. Hence, proved.

10. $\frac{\tan 5\theta+\tan 3\theta}{\tan 5\theta-\tan 3\theta} = 4\cos 2\theta\cos 4\theta$

Solution:
Convert tan to sin/cos on the LHS.

$$\text{LHS} = \frac{\frac{\sin 5\theta}{\cos 5\theta}+\frac{\sin 3\theta}{\cos 3\theta}}{\frac{\sin 5\theta}{\cos 5\theta}-\frac{\sin 3\theta}{\cos 3\theta}} = \frac{\sin 5\theta\cos 3\theta+\cos 5\theta\sin 3\theta}{\sin 5\theta\cos 3\theta-\cos 5\theta\sin 3\theta}$$
$$= \frac{\sin(5\theta+3\theta)}{\sin(5\theta-3\theta)} = \frac{\sin 8\theta}{\sin 2\theta}$$
$$= \frac{2\sin 4\theta\cos 4\theta}{\sin 2\theta} \qquad [\text{Using } \sin 2X = 2\sin X\cos X]$$
$$= \frac{2(2\sin 2\theta\cos 2\theta)\cos 4\theta}{\sin 2\theta}$$
$$= 4\cos 2\theta\cos 4\theta.$$

LHS = RHS. Hence, proved.

11. $\frac{\cos 3\theta+2\cos 5\theta+\cos 7\theta}{\cos\theta+2\cos 3\theta+\cos 5\theta} = \cos 2\theta - \sin 2\theta\tan 3\theta$

Solution:
First, we simplify the Left Hand Side (LHS) by grouping terms and using the sum-to-product formula, $\cos A+\cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$.

$$\text{LHS} = \frac{(\cos 7\theta+\cos 3\theta)+2\cos 5\theta}{(\cos 5\theta+\cos\theta)+2\cos 3\theta}$$
$$= \frac{2\cos\left(\frac{7\theta+3\theta}{2}\right)\cos\left(\frac{7\theta-3\theta}{2}\right)+2\cos 5\theta}{2\cos\left(\frac{5\theta+\theta}{2}\right)\cos\left(\frac{5\theta-\theta}{2}\right)+2\cos 3\theta}$$
$$= \frac{2\cos 5\theta\cos 2\theta+2\cos 5\theta}{2\cos 3\theta\cos 2\theta+2\cos 3\theta}$$
$$= \frac{2\cos 5\theta(\cos 2\theta+1)}{2\cos 3\theta(\cos 2\theta+1)} \qquad [\text{Factoring out common terms}]$$
$$= \frac{\cos 5\theta}{\cos 3\theta}.$$

Now, we will show that this simplified expression is equal to the Right Hand Side (RHS). We rewrite

$\cos 5\theta$ as $\cos(3\theta + 2\theta)$ and use the cosine sum identity.

$$\begin{aligned}
\frac{\cos 5\theta}{\cos 3\theta} &= \frac{\cos(3\theta + 2\theta)}{\cos 3\theta} \\
&= \frac{\cos 3\theta \cos 2\theta - \sin 3\theta \sin 2\theta}{\cos 3\theta} && \text{[Using } \cos(A+B) \text{ formula]} \\
&= \frac{\cos 3\theta \cos 2\theta}{\cos 3\theta} - \frac{\sin 3\theta \sin 2\theta}{\cos 3\theta} && \text{[Splitting the fraction]} \\
&= \cos 2\theta - \left(\frac{\sin 3\theta}{\cos 3\theta}\right) \sin 2\theta \\
&= \cos 2\theta - \tan 3\theta \sin 2\theta.
\end{aligned}$$

Thus, we have shown that the LHS simplifies to the RHS. Hence, proved.

12. $\frac{\sin A + \sin 3A + \sin 5A + \sin 7A}{\cos A + \cos 3A + \cos 5A + \cos 7A} = \tan 4A$

Solution:
Group terms on the LHS and use sum-to-product.

$$\begin{aligned}
\text{LHS} &= \frac{(\sin 7A + \sin A) + (\sin 5A + \sin 3A)}{(\cos 7A + \cos A) + (\cos 5A + \cos 3A)} \\
&= \frac{2 \sin 4A \cos 3A + 2 \sin 4A \cos A}{2 \cos 4A \cos 3A + 2 \cos 4A \cos A} \\
&= \frac{2 \sin 4A(\cos 3A + \cos A)}{2 \cos 4A(\cos 3A + \cos A)} = \frac{\sin 4A}{\cos 4A} = \tan 4A.
\end{aligned}$$

LHS = RHS. Hence, proved.

13. $\frac{\sin(\theta+\phi) - 2\sin\theta + \sin(\theta-\phi)}{\cos(\theta+\phi) - 2\cos\theta + \cos(\theta-\phi)} = \tan\theta$

Solution:
Group terms on the LHS and use sum-to-product.

$$\begin{aligned}
\text{LHS} &= \frac{(\sin(\theta+\phi) + \sin(\theta-\phi)) - 2\sin\theta}{(\cos(\theta+\phi) + \cos(\theta-\phi)) - 2\cos\theta} \\
&= \frac{2\sin\theta\cos\phi - 2\sin\theta}{2\cos\theta\cos\phi - 2\cos\theta} \\
&= \frac{2\sin\theta(\cos\phi - 1)}{2\cos\theta(\cos\phi - 1)} = \frac{\sin\theta}{\cos\theta} = \tan\theta.
\end{aligned}$$

LHS = RHS. Hence, proved.

14. $\frac{\sin A - \sin 5A + \sin 9A - \sin 13A}{\cos A - \cos 5A - \cos 9A + \cos 13A} = \cot 4A$

Solution:
Group terms on the LHS.

$$\begin{aligned}
\text{Numerator} &= (\sin 9A - \sin 13A) - (\sin 5A - \sin A) \\
&= 2\cos\left(\frac{22A}{2}\right)\sin\left(\frac{-4A}{2}\right) - 2\cos\left(\frac{6A}{2}\right)\sin\left(\frac{4A}{2}\right) \\
&= 2\cos 11A(-\sin 2A) - 2\cos 3A \sin 2A = -2\sin 2A(\cos 11A + \cos 3A) \\
&= -2\sin 2A(2\cos 7A \cos 4A) = -4\sin 2A \cos 4A \cos 7A.
\end{aligned}$$

$$\begin{aligned}\text{Denominator} &= (\cos 13A + \cos A) - (\cos 5A + \cos 9A) \\ &= 2\cos 7A\cos 6A - 2\cos 7A\cos 2A = 2\cos 7A(\cos 6A - \cos 2A) \\ &= 2\cos 7A(-2\sin 4A\sin 2A) = -4\cos 7A\sin 4A\sin 2A.\end{aligned}$$

$$\text{LHS} = \frac{-4\sin 2A\cos 4A\cos 7A}{-4\cos 7A\sin 4A\sin 2A} = \frac{\cos 4A}{\sin 4A} = \cot 4A.$$

LHS = RHS. Hence, proved.

15. $\frac{\sin A+\sin B}{\sin A-\sin B} = \tan\frac{A+B}{2}\cot\frac{A-B}{2}$

Solution:

Using sum-to-product formulas on the LHS:

$$\begin{aligned}\text{LHS} &= \frac{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}{2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)} \\ &= \frac{\sin((A+B)/2)}{\cos((A+B)/2)}\cdot\frac{\cos((A-B)/2)}{\sin((A-B)/2)} = \tan\frac{A+B}{2}\cot\frac{A-B}{2}.\end{aligned}$$

LHS = RHS. Hence, proved.

16. $\frac{\sin A-\sin B}{\cos B-\cos A} = \cot\frac{A+B}{2}$

Solution:

Using sum-to-product formulas on the LHS:

$$\begin{aligned}\text{LHS} &= \frac{2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}{-2\sin\left(\frac{B+A}{2}\right)\sin\left(\frac{B-A}{2}\right)} \\ &= \frac{\cos((A+B)/2)\sin((A-B)/2)}{-\sin((A+B)/2)[-\sin((A-B)/2)]} = \frac{\cos((A+B)/2)}{\sin((A+B)/2)} = \cot\frac{A+B}{2}.\end{aligned}$$

LHS = RHS. Hence, proved.

17. $\frac{\cos(A+B+C)+\cos(-A+B+C)+\cos(A-B+C)+\cos(A+B-C)}{\sin(A+B+C)+\sin(-A+B+C)-\sin(A-B+C)+\sin(A+B-C)} = \cot B$

Solution:

Let's group terms and apply sum-to-product formulas.

$$\begin{aligned}\text{Num} &= [\cos(B+C+A)+\cos(B+C-A)] + [\cos(A+(B-C))+\cos(A-(B-C))] \\ &= 2\cos(B+C)\cos A + 2\cos A\cos(B-C) \\ &= 2\cos A(\cos(B+C)+\cos(B-C)) = 2\cos A(2\cos B\cos C) = 4\cos A\cos B\cos C.\end{aligned}$$

$$\begin{aligned}\text{Den} &= [\sin(A+B+C)+\sin(-A+B+C)] + [\sin(A+B-C)-\sin(A-B+C)] \\ &= 2\sin(B+C)\cos A + 2\cos(A)\sin(B-C) \\ &= 2\cos A(\sin(B+C)+\sin(B-C)) = 2\cos A(2\sin B\cos C) = 4\cos A\sin B\cos C.\end{aligned}$$

$$\text{LHS} = \frac{4\cos A\cos B\cos C}{4\cos A\sin B\cos C} = \frac{\cos B}{\sin B} = \cot B.$$

LHS = RHS. Hence, proved.

18. $\cos 3A + \cos 5A + \cos 7A + \cos 15A = 4\cos 4A \cos 5A \cos 6A$

Solution:

Group terms on the LHS.

$$\begin{aligned}
\text{LHS} &= (\cos 15A + \cos 3A) + (\cos 7A + \cos 5A) \\
&= 2\cos\left(\frac{18A}{2}\right)\cos\left(\frac{12A}{2}\right) + 2\cos\left(\frac{12A}{2}\right)\cos\left(\frac{2A}{2}\right) \\
&= 2\cos 9A \cos 6A + 2\cos 6A \cos A = 2\cos 6A(\cos 9A + \cos A) \\
&= 2\cos 6A(2\cos 5A \cos 4A) = 4\cos 4A \cos 5A \cos 6A.
\end{aligned}$$

LHS = RHS. Hence, proved.

19. $\cos(-A + B + C) + \cos(A - B + C) + \cos(A + B - C) + \cos(A + B + C) = 4\cos A \cos B \cos C$

Solution:

This is the same as the numerator of question 17.

$$\begin{aligned}
\text{LHS} &= [\cos(B + C - A) + \cos(B + C + A)] + [\cos(A - (B - C)) + \cos(A + (B - C))] \\
&= 2\cos(B + C)\cos(-A) + 2\cos A \cos(B - C) \\
&= 2\cos A(\cos(B + C) + \cos(B - C)) \\
&= 2\cos A(2\cos B \cos C) = 4\cos A \cos B \cos C.
\end{aligned}$$

LHS = RHS. Hence, proved.

20. $\sin 50^\circ - \sin 70^\circ + \sin 10^\circ = 0$

Solution:

Group the first two terms on the LHS.

$$\begin{aligned}
\text{LHS} &= (\sin 50^\circ - \sin 70^\circ) + \sin 10^\circ \\
&= 2\cos\left(\frac{50 + 70}{2}\right)\sin\left(\frac{50 - 70}{2}\right) + \sin 10^\circ \\
&= 2\cos(60^\circ)\sin(-10^\circ) + \sin 10^\circ \\
&= 2\left(\frac{1}{2}\right)(-\sin 10^\circ) + \sin 10^\circ = -\sin 10^\circ + \sin 10^\circ = 0.
\end{aligned}$$

LHS = RHS. Hence, proved.

21. $\sin 10^\circ + \sin 20^\circ + \sin 40^\circ + \sin 50^\circ = \sin 70^\circ + \sin 80^\circ$

Solution:

Let's work on the LHS by grouping terms.

$$\begin{aligned}
\text{LHS} &= (\sin 50^\circ + \sin 10^\circ) + (\sin 40^\circ + \sin 20^\circ) \\
&= 2\sin\left(\frac{50 + 10}{2}\right)\cos\left(\frac{50 - 10}{2}\right) + 2\sin\left(\frac{40 + 20}{2}\right)\cos\left(\frac{40 - 20}{2}\right) \\
&= 2\sin 30^\circ \cos 20^\circ + 2\sin 30^\circ \cos 10^\circ \\
&= 2\left(\frac{1}{2}\right)\cos 20^\circ + 2\left(\frac{1}{2}\right)\cos 10^\circ = \cos 20^\circ + \cos 10^\circ.
\end{aligned}$$

Now, let's work on the RHS using co-function identities.

$$\text{RHS} = \sin 70^\circ + \sin 80^\circ = \sin(90^\circ - 20^\circ) + \sin(90^\circ - 10^\circ) = \cos 20^\circ + \cos 10^\circ.$$

LHS = RHS. Hence, proved.

22. $\sin\alpha + \sin 2\alpha + \sin 4\alpha + \sin 5\alpha = 4\cos\frac{\alpha}{2}\cos\frac{3\alpha}{2}\sin 3\alpha$

Solution:
Group terms on the LHS.

$$\begin{aligned}
\text{LHS} &= (\sin 5\alpha + \sin\alpha) + (\sin 4\alpha + \sin 2\alpha) \\
&= 2\sin\left(\frac{6\alpha}{2}\right)\cos\left(\frac{4\alpha}{2}\right) + 2\sin\left(\frac{6\alpha}{2}\right)\cos\left(\frac{2\alpha}{2}\right) \\
&= 2\sin 3\alpha\cos 2\alpha + 2\sin 3\alpha\cos\alpha \\
&= 2\sin 3\alpha(\cos 2\alpha + \cos\alpha) \\
&= 2\sin 3\alpha\left(2\cos\frac{3\alpha}{2}\cos\frac{\alpha}{2}\right) \\
&= 4\cos\frac{\alpha}{2}\cos\frac{3\alpha}{2}\sin 3\alpha.
\end{aligned}$$

LHS = RHS. Hence, proved.

Que.3: SL LONEY Ex.15

1-4. Express as a sum or difference the following:

Solution:
We use the product-to-sum formulas.

1. $2\sin 5\theta \sin 7\theta = \cos(7\theta - 5\theta) - \cos(7\theta + 5\theta) = \cos \mathbf{2}\theta - \cos \mathbf{12}\theta$.

2. $2\cos 7\theta \sin 5\theta = \sin(7\theta + 5\theta) - \sin(7\theta - 5\theta) = \sin \mathbf{12}\theta - \sin \mathbf{2}\theta$.

3. $2\cos 11\theta \cos 3\theta = \cos(11\theta + 3\theta) + \cos(11\theta - 3\theta) = \cos \mathbf{14}\theta + \cos \mathbf{8}\theta$.

4. $2\sin 54^\circ \sin 66^\circ = \cos(66^\circ - 54^\circ) - \cos(66^\circ + 54^\circ) = \cos \mathbf{12}^\circ - \cos \mathbf{120}^\circ$.

5. Prove that: $\sin\frac{\theta}{2}\sin\frac{7\theta}{2} + \sin\frac{3\theta}{2}\sin\frac{11\theta}{2} = \sin 2\theta \sin 5\theta$

Solution:
Multiply the LHS by 2 and divide by 2, then apply product-to-sum.

$$\begin{aligned}
\text{LHS} &= \frac{1}{2}\left[2\sin\frac{7\theta}{2}\sin\frac{\theta}{2} + 2\sin\frac{11\theta}{2}\sin\frac{3\theta}{2}\right] \\
&= \frac{1}{2}\left[\left(\cos\left(\frac{6\theta}{2}\right) - \cos\left(\frac{8\theta}{2}\right)\right) + \left(\cos\left(\frac{8\theta}{2}\right) - \cos\left(\frac{14\theta}{2}\right)\right)\right] \\
&= \frac{1}{2}\left[\cos 3\theta - \cos 4\theta + \cos 4\theta - \cos 7\theta\right] \\
&= \frac{1}{2}(\cos 3\theta - \cos 7\theta) = \frac{1}{2}\left(-2\sin\left(\frac{10\theta}{2}\right)\sin\left(\frac{-4\theta}{2}\right)\right) = \sin 5\theta \sin 2\theta.
\end{aligned}$$

LHS = RHS. Hence, proved.

6. Prove that: $\cos 2\theta \cos\frac{\theta}{2} - \cos 3\theta \cos\frac{9\theta}{2} = \sin 5\theta \sin\frac{5\theta}{2}$

Solution:
Multiply the LHS by 2 and divide by 2, then apply product-to-sum.

$$\begin{aligned}
\text{LHS} &= \frac{1}{2}\left[2\cos 2\theta \cos\frac{\theta}{2} - 2\cos 3\theta \cos\frac{9\theta}{2}\right] \\
&= \frac{1}{2}\left[\left(\cos\frac{5\theta}{2} + \cos\frac{3\theta}{2}\right) - \left(\cos\frac{15\theta}{2} + \cos\frac{3\theta}{2}\right)\right] \\
&= \frac{1}{2}\left(\cos\frac{5\theta}{2} - \cos\frac{15\theta}{2}\right) = \frac{1}{2}\left(-2\sin\left(\frac{20\theta}{4}\right)\sin\left(\frac{-10\theta}{4}\right)\right) \\
&= \sin(5\theta)\sin(5\theta/2).
\end{aligned}$$

LHS = RHS. Hence, proved.

7. Prove that: $\sin A \sin(A + 2B) - \sin B \sin(B + 2A) = \sin(A - B)\sin(A + B)$

Solution:

Multiply the LHS by 2 and divide by 2, then apply product-to-sum.

$$\begin{aligned}\text{LHS} &= \frac{1}{2}[2\sin(A+2B)\sin A - 2\sin(B+2A)\sin B] \\ &= \frac{1}{2}[(\cos(2B) - \cos(2A+2B)) - (\cos(2A) - \cos(2B+2A))] \\ &= \frac{1}{2}[\cos 2B - \cos(2A+2B) - \cos 2A + \cos(2A+2B)] \\ &= \frac{1}{2}(\cos 2B - \cos 2A) = \frac{1}{2}(-2\sin(B+A)\sin(B-A)) \\ &= \sin(A+B)\sin(A-B).\end{aligned}$$

LHS = RHS. Hence, proved.

8. Prove that: $(\sin 3A + \sin A)\sin A + (\cos 3A - \cos A)\cos A = 0$

Solution:
Apply sum-to-product on the terms in parentheses.

$$\begin{aligned}\text{LHS} &= (2\sin 2A\cos A)\sin A + (-2\sin 2A\sin A)\cos A \\ &= 2\sin 2A\cos A\sin A - 2\sin 2A\sin A\cos A = 0.\end{aligned}$$

LHS = RHS. Hence, proved.

9. $\frac{2\sin(A-C)\cos C - \sin(A-2C)}{2\sin(B-C)\cos C - \sin(B-2C)} = \frac{\sin A}{\sin B}$

Solution:
Apply product-to-sum on the first term in the numerator and denominator.

$$\begin{aligned}\text{Numerator} &= [\sin((A-C)+C) + \sin((A-C)-C)] - \sin(A-2C) \\ &= \sin A + \sin(A-2C) - \sin(A-2C) = \sin A.\end{aligned}$$

$$\begin{aligned}\text{Denominator} &= [\sin((B-C)+C) + \sin((B-C)-C)] - \sin(B-2C) \\ &= \sin B + \sin(B-2C) - \sin(B-2C) = \sin B.\end{aligned}$$

LHS $= \frac{\sin A}{\sin B}$. Hence, proved.

10. $\frac{\sin A\sin 2A + \sin 3A\sin 6A + \sin 4A\sin 13A}{\sin A\cos 2A + \sin 3A\cos 6A + \sin 4A\cos 13A} = \tan 9A$

Solution:
Multiply the numerator and denominator by 2 and apply product-to-sum/difference formulas.

$$\begin{aligned}\text{Numerator} &= (2\sin A\sin 2A) + (2\sin 3A\sin 6A) + (2\sin 4A\sin 13A) \\ &= (\cos A - \cos 3A) + (\cos 3A - \cos 9A) + (\cos 9A - \cos 17A) \\ &= \cos A - \cos 17A.\end{aligned}$$

$$\begin{aligned}\text{Denominator} &= (2\sin A\cos 2A) + (2\sin 3A\cos 6A) + (2\sin 4A\cos 13A) \\ &= (\sin 3A + \sin(-A)) + (\sin 9A + \sin(-3A)) + (\sin 17A + \sin(-9A)) \\ &= (\sin 3A - \sin A) + (\sin 9A - \sin 3A) + (\sin 17A - \sin 9A) = \sin 17A - \sin A.\end{aligned}$$

Now, combine the simplified numerator and denominator.

$$\text{LHS} = \frac{\cos A - \cos 17A}{\sin 17A - \sin A} = \frac{-2\sin\left(\frac{18A}{2}\right)\sin\left(\frac{-16A}{2}\right)}{2\cos\left(\frac{18A}{2}\right)\sin\left(\frac{16A}{2}\right)} = \frac{2\sin 9A\sin 8A}{2\cos 9A\sin 8A} = \tan 9A.$$

LHS = RHS. Hence, proved.

11. $\cos(36^\circ - A)\cos(36^\circ + A) + \cos(54^\circ + A)\cos(54^\circ - A) = \cos 2A$

Solution:
Use the identity $\cos(X - Y)\cos(X + Y) = \cos^2 X - \sin^2 Y$.

$$\begin{aligned} \text{LHS} &= (\cos^2 36^\circ - \sin^2 A) + (\cos^2 54^\circ - \sin^2 A) \\ &= \cos^2 36^\circ + \cos^2(90^\circ - 36^\circ) - 2\sin^2 A \\ &= \cos^2 36^\circ + \sin^2 36^\circ - 2\sin^2 A = 1 - 2\sin^2 A = \cos 2A. \end{aligned}$$

LHS = RHS. Hence, proved.

12. $\cos A \sin(B - C) + \cos B \sin(C - A) + \cos C \sin(A - B) = 0$

Solution:
Expand the sine terms on the LHS.

$$\begin{aligned} \text{LHS} &= \cos A(\sin B \cos C - \cos B \sin C) + \cos B(\sin C \cos A - \cos C \sin A) \\ &\quad + \cos C(\sin A \cos B - \cos A \sin B) \\ &= \cos A \sin B \cos C - \cos A \cos B \sin C + \cos B \sin C \cos A - \cos B \cos C \sin A \\ &\quad + \cos C \sin A \cos B - \cos C \cos A \sin B \\ &= 0 \end{aligned}$$

[All terms cancel out in pairs. For example, the first term cancels with the sixth.]

LHS = RHS. Hence, proved.

13. $\sin(45^\circ + A)\sin(45^\circ - A) = \frac{1}{2}\cos 2A$

Solution:
Use the identity $\sin(X + Y)\sin(X - Y) = \sin^2 X - \sin^2 Y$.

$$\begin{aligned} \text{LHS} &= \sin^2 45^\circ - \sin^2 A = \left(\frac{1}{\sqrt{2}}\right)^2 - \sin^2 A \\ &= \frac{1}{2} - \sin^2 A = \frac{1 - 2\sin^2 A}{2} = \frac{\cos 2A}{2}. \end{aligned}$$

LHS = RHS. Hence, proved.

14. $\sin(\beta - \gamma)\cos(\alpha - \delta) + \sin(\gamma - \alpha)\cos(\beta - \delta) + \sin(\alpha - \beta)\cos(\gamma - \delta) = 0$

Solution:
Multiply the entire LHS by 2 and use the formula $2\sin X \cos Y = \sin(X + Y) + \sin(X - Y)$.

$$\begin{aligned} 2 \times \text{LHS} &= [2\sin(\beta - \gamma)\cos(\alpha - \delta)] + [2\sin(\gamma - \alpha)\cos(\beta - \delta)] + [2\sin(\alpha - \beta)\cos(\gamma - \delta)] \\ &= [\sin(\beta - \gamma + \alpha - \delta) + \sin(\beta - \gamma - \alpha + \delta)] \\ &\quad + [\sin(\gamma - \alpha + \beta - \delta) + \sin(\gamma - \alpha - \beta + \delta)] \\ &\quad + [\sin(\alpha - \beta + \gamma - \delta) + \sin(\alpha - \beta - \gamma + \delta)] \end{aligned}$$

Observe that terms cancel in pairs. For example, $\sin(\beta - \gamma - \alpha + \delta) = -\sin(\alpha - \beta + \gamma - \delta)$. All terms will cancel.

$$2 \times \text{LHS} = 0 \implies \text{LHS} = 0.$$

Hence, proved.

15. $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}=0$

Solution:

Apply the product-to-sum formula $2\cos X\cos Y=\cos(X+Y)+\cos(X-Y)$ to the first term of the LHS.

$$\begin{aligned}\text{LHS} &= \left[\cos\left(\frac{\pi}{13}+\frac{9\pi}{13}\right)+\cos\left(\frac{9\pi}{13}-\frac{\pi}{13}\right)\right]+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}\\ &= \cos\frac{10\pi}{13}+\cos\frac{8\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}\end{aligned}$$

Now, use the identity $\cos x=-\cos(\pi-x)$.

$$\begin{aligned}\cos\frac{10\pi}{13} &= \cos\left(\pi-\frac{3\pi}{13}\right)=-\cos\frac{3\pi}{13}.\\ \cos\frac{8\pi}{13} &= \cos\left(\pi-\frac{5\pi}{13}\right)=-\cos\frac{5\pi}{13}.\end{aligned}$$

Substituting these back into the expression:

$$\text{LHS}=\left(-\cos\frac{3\pi}{13}\right)+\left(-\cos\frac{5\pi}{13}\right)+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}=0.$$

LHS = RHS. Hence, proved.

Que 4: SL LONEY EX.17

1. Find the value of $\sin 2\alpha$ when

(i) $\cos\alpha = \frac{3}{5}$

(ii) $\sin\alpha = \frac{12}{13}$

(iii) $\tan\alpha = \frac{16}{63}$

Solution:
We use the formula $\sin 2\alpha = 2\sin\alpha\cos\alpha$. We will assume α is in the first quadrant, so all trigonometric ratios are positive.

(i) Given $\cos\alpha = 3/5$. First, find $\sin\alpha$:

$$\sin\alpha = \sqrt{1-\cos^2\alpha} = \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}.$$

Now, calculate $\sin 2\alpha$:

$$\sin 2\alpha = 2\sin\alpha\cos\alpha = 2\left(\frac{4}{5}\right)\left(\frac{3}{5}\right) = \mathbf{\frac{24}{25}}.$$

(ii) Given $\sin\alpha = 12/13$. First, find $\cos\alpha$:

$$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\left(\frac{12}{13}\right)^2} = \sqrt{1-\frac{144}{169}} = \sqrt{\frac{25}{169}} = \frac{5}{13}.$$

Now, calculate $\sin 2\alpha$:

$$\sin 2\alpha = 2\sin\alpha\cos\alpha = 2\left(\frac{12}{13}\right)\left(\frac{5}{13}\right) = \mathbf{\frac{120}{169}}.$$

(iii) Given $\tan\alpha = 16/63$. We can use the formula $\sin 2\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha}$.

$$\sin 2\alpha = \frac{2(16/63)}{1+(16/63)^2} = \frac{32/63}{1+256/3969} = \frac{32/63}{\frac{3969+256}{3969}} = \frac{32/63}{4225/3969}$$
$$= \frac{32}{63}\times\frac{3969}{4225} = 32\times\frac{63}{4225} = \mathbf{\frac{2016}{4225}}.$$

2. Find the value of $\cos 2\alpha$ when

(i) $\cos\alpha = \frac{15}{17}$

(ii) $\sin\alpha = \frac{4}{5}$

(iii) $\tan\alpha = \frac{5}{12}$

Solution:

(i) Using the formula $\cos 2\alpha = 2\cos^2\alpha - 1$.

$$\cos 2\alpha = 2\left(\frac{15}{17}\right)^2 - 1 = 2\left(\frac{225}{289}\right) - 1 = \frac{450-289}{289} = \mathbf{\frac{161}{289}}.$$

(ii) Using the formula $\cos 2\alpha = 1 - 2\sin^2\alpha$.

$$\cos 2\alpha = 1 - 2\left(\frac{4}{5}\right)^2 = 1 - 2\left(\frac{16}{25}\right) = 1 - \frac{32}{25} = -\frac{\mathbf{7}}{\mathbf{25}}.$$

(iii) Using the formula $\cos 2\alpha = \frac{1-\tan^2\alpha}{1+\tan^2\alpha}$.

$$\cos 2\alpha = \frac{1-(5/12)^2}{1+(5/12)^2} = \frac{1-25/144}{1+25/144} = \frac{(144-25)/144}{(144+25)/144} = \frac{\mathbf{119}}{\mathbf{169}}.$$

3. If $\tan\theta = \frac{b}{a}$, find the value of $a\cos 2\theta + b\sin 2\theta$.

Solution:

We express $\cos 2\theta$ and $\sin 2\theta$ using the t-formulas (in terms of $\tan\theta$).

$$\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-(b/a)^2}{1+(b/a)^2} = \frac{\frac{a^2-b^2}{a^2}}{\frac{a^2+b^2}{a^2}} = \frac{a^2-b^2}{a^2+b^2}.$$

$$\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta} = \frac{2(b/a)}{1+(b/a)^2} = \frac{2b/a}{\frac{a^2+b^2}{a^2}} = \frac{2ab}{a^2+b^2}.$$

Now substitute these into the given expression.

$$\begin{aligned} a\cos 2\theta + b\sin 2\theta &= a\left(\frac{a^2-b^2}{a^2+b^2}\right) + b\left(\frac{2ab}{a^2+b^2}\right) \\ &= \frac{a(a^2-b^2)+b(2ab)}{a^2+b^2} = \frac{a^3-ab^2+2ab^2}{a^2+b^2} \\ &= \frac{a^3+ab^2}{a^2+b^2} = \frac{a(a^2+b^2)}{a^2+b^2} = \mathbf{a}. \end{aligned}$$

4. Prove that: $\frac{\sin 2A}{1+\cos 2A} = \tan A$

Solution:

We use the double angle formulas for sine and cosine on the LHS.

$$\text{LHS} = \frac{2\sin A\cos A}{1+(2\cos^2 A - 1)} = \frac{2\sin A\cos A}{2\cos^2 A} = \frac{\sin A}{\cos A} = \tan A.$$

LHS = RHS. Hence, proved.

5. Prove that: $\frac{\sin 2A}{1-\cos 2A} = \cot A$

Solution:

We use the double angle formulas for sine and cosine on the LHS.

$$\text{LHS} = \frac{2\sin A\cos A}{1-(1-2\sin^2 A)} = \frac{2\sin A\cos A}{2\sin^2 A} = \frac{\cos A}{\sin A} = \cot A.$$

LHS = RHS. Hence, proved.

6. Prove that: $\frac{1-\cos 2A}{1+\cos 2A} = \tan^2 A$

Solution:
We use the double angle formulas for cosine on the LHS.

$$\text{LHS} = \frac{1-(1-2\sin^2 A)}{1+(2\cos^2 A-1)} = \frac{2\sin^2 A}{2\cos^2 A} = \tan^2 A.$$

LHS = RHS. Hence, proved.

7. Prove that: $\tan A + \cot A = 2\,\text{cosec}\,2A$

Solution:
We express tan and cot in terms of sin and cos.

$$\begin{aligned}\text{LHS} &= \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A} = \frac{\sin^2 A + \cos^2 A}{\sin A\cos A} \\ &= \frac{1}{\sin A\cos A} && [\text{Using } \sin^2 A + \cos^2 A = 1] \\ &= \frac{2}{2\sin A\cos A} && [\text{Multiply numerator and denominator by 2}] \\ &= \frac{2}{\sin 2A} = 2\,\text{cosec}\,2A.\end{aligned}$$

LHS = RHS. Hence, proved.

8. Prove that: $\tan A - \cot A = -2\cot 2A$

Solution:
We express tan and cot in terms of sin and cos.

$$\begin{aligned}\text{LHS} &= \frac{\sin A}{\cos A} - \frac{\cos A}{\sin A} = \frac{\sin^2 A - \cos^2 A}{\sin A\cos A} \\ &= \frac{-(\cos^2 A - \sin^2 A)}{\frac{1}{2}(2\sin A\cos A)} = \frac{-\cos 2A}{\frac{1}{2}\sin 2A} = -2\frac{\cos 2A}{\sin 2A} = -2\cot 2A.\end{aligned}$$

LHS = RHS. Hence, proved.

9. Prove that: $\text{cosec}\,2A + \cot 2A = \cot A$

Solution:
We express csc and cot in terms of sin and cos.

$$\begin{aligned}\text{LHS} &= \frac{1}{\sin 2A} + \frac{\cos 2A}{\sin 2A} = \frac{1+\cos 2A}{\sin 2A} \\ &= \frac{1+(2\cos^2 A-1)}{2\sin A\cos A} = \frac{2\cos^2 A}{2\sin A\cos A} = \frac{\cos A}{\sin A} = \cot A.\end{aligned}$$

LHS = RHS. Hence, proved.

10. Prove that: $\frac{\cos A}{1+\sin A} = \tan(45^\circ - \frac{A}{2})$

Solution:

We use half-angle formulas for the LHS.

$$\begin{aligned}
\text{LHS} &= \frac{\cos^2(A/2) - \sin^2(A/2)}{1 + 2\sin(A/2)\cos(A/2)} \\
&= \frac{\cos^2(A/2) - \sin^2(A/2)}{\cos^2(A/2) + \sin^2(A/2) + 2\sin(A/2)\cos(A/2)} \quad [\text{Using } 1 = \cos^2\theta + \sin^2\theta] \\
&= \frac{(\cos(A/2) - \sin(A/2))(\cos(A/2) + \sin(A/2))}{(\cos(A/2) + \sin(A/2))^2} \\
&= \frac{\cos(A/2) - \sin(A/2)}{\cos(A/2) + \sin(A/2)}
\end{aligned}$$

Now, divide the numerator and denominator by $\cos(A/2)$.

$$= \frac{1 - \tan(A/2)}{1 + \tan(A/2)} = \frac{\tan 45^\circ - \tan(A/2)}{1 + \tan 45^\circ \tan(A/2)} = \tan(45^\circ - A/2).$$

LHS = RHS. Hence, proved.

11. Prove that: $\frac{\sec 8A - 1}{\sec 4A - 1} = \frac{\tan 8A}{\tan 2A}$

Solution:
We convert sec to cos.

$$\begin{aligned}
\text{LHS} &= \frac{\frac{1}{\cos 8A} - 1}{\frac{1}{\cos 4A} - 1} = \frac{\frac{1-\cos 8A}{\cos 8A}}{\frac{1-\cos 4A}{\cos 4A}} = \frac{1 - \cos 8A}{1 - \cos 4A} \cdot \frac{\cos 4A}{\cos 8A} \\
&= \frac{2\sin^2 4A}{2\sin^2 2A} \cdot \frac{\cos 4A}{\cos 8A} \quad [\text{Using } 1 - \cos 2X = 2\sin^2 X] \\
&= \frac{\sin 4A \cdot (2\sin 4A \cos 4A)}{2\sin^2 2A \cos 8A} = \frac{\sin 4A \cdot \sin 8A}{2\sin^2 2A \cos 8A} \\
&= \frac{(2\sin 2A \cos 2A)\sin 8A}{2\sin^2 2A \cos 8A} = \frac{\cos 2A \sin 8A}{\sin 2A \cos 8A} = \cot 2A \tan 8A = \frac{\tan 8A}{\tan 2A}.
\end{aligned}$$

LHS = RHS. Hence, proved.

12. Prove that: $\frac{1+\tan^2(45^\circ - A)}{1-\tan^2(45^\circ - A)} = \operatorname{cosec} 2A$

Solution:
Let $\theta = 45^\circ - A$. The LHS is in the form $\frac{1+\tan^2\theta}{1-\tan^2\theta}$.

$$\begin{aligned}
\text{LHS} &= \frac{1 + \tan^2\theta}{1 - \tan^2\theta} = \frac{1}{\frac{1-\tan^2\theta}{1+\tan^2\theta}} = \frac{1}{\cos 2\theta} \\
&= \frac{1}{\cos(2(45^\circ - A))} = \frac{1}{\cos(90^\circ - 2A)} \\
&= \frac{1}{\sin 2A} = \operatorname{cosec} 2A.
\end{aligned}$$

LHS = RHS. Hence, proved.

13. Prove that: $\frac{\sin\alpha + \sin\beta}{\sin\alpha - \sin\beta} = \frac{\tan\frac{\alpha+\beta}{2}}{\tan\frac{\alpha-\beta}{2}}$

Solution:

We use the sum-to-product formulas on the LHS.

$$\begin{aligned}
\text{LHS} &= \frac{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}{2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)} \\
&= \frac{\sin((\alpha+\beta)/2)}{\cos((\alpha+\beta)/2)} \cdot \frac{\cos((\alpha-\beta)/2)}{\sin((\alpha-\beta)/2)} \\
&= \tan\frac{\alpha+\beta}{2}\cot\frac{\alpha-\beta}{2} = \frac{\tan\frac{\alpha+\beta}{2}}{\tan\frac{\alpha-\beta}{2}}.
\end{aligned}$$

LHS = RHS. Hence, proved.

14. Prove that: $\frac{\sin^2 A - \sin^2 B}{\sin A\cos A - \sin B\cos B} = \tan(A+B)$

Solution:

We use the identity $\sin^2 X - \sin^2 Y = \sin(X+Y)\sin(X-Y)$ in the numerator. In the denominator, we use the identity $\sin X\cos X = \frac{1}{2}\sin 2X$.

$$\text{LHS} = \frac{\sin(A+B)\sin(A-B)}{\frac{1}{2}\sin 2A - \frac{1}{2}\sin 2B} = \frac{2\sin(A+B)\sin(A-B)}{\sin 2A - \sin 2B}$$

Now apply sum-to-product on the denominator.

$$\begin{aligned}
&= \frac{2\sin(A+B)\sin(A-B)}{2\cos\left(\frac{2A+2B}{2}\right)\sin\left(\frac{2A-2B}{2}\right)} \\
&= \frac{2\sin(A+B)\sin(A-B)}{2\cos(A+B)\sin(A-B)} = \frac{\sin(A+B)}{\cos(A+B)} = \tan(A+B).
\end{aligned}$$

LHS = RHS. Hence, proved.

15. Prove that: $\tan(\frac{\pi}{4}+\theta) - \tan(\frac{\pi}{4}-\theta) = 2\tan 2\theta$

Solution:

We use the $\tan(X \pm Y)$ formulas.

$$\begin{aligned}
\text{LHS} &= \frac{\tan(\pi/4)+\tan\theta}{1-\tan(\pi/4)\tan\theta} - \frac{\tan(\pi/4)-\tan\theta}{1+\tan(\pi/4)\tan\theta} \\
&= \frac{1+\tan\theta}{1-\tan\theta} - \frac{1-\tan\theta}{1+\tan\theta} \\
&= \frac{(1+\tan\theta)^2 - (1-\tan\theta)^2}{(1-\tan\theta)(1+\tan\theta)} \\
&= \frac{(1+2\tan\theta+\tan^2\theta) - (1-2\tan\theta+\tan^2\theta)}{1-\tan^2\theta} \\
&= \frac{4\tan\theta}{1-\tan^2\theta} = 2\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = 2\tan 2\theta.
\end{aligned}$$

LHS = RHS. Hence, proved.

16. Prove that: $\frac{\cos A+\sin A}{\cos A-\sin A} - \frac{\cos A-\sin A}{\cos A+sinA} = 2\tan 2A$

Solution:

We find a common denominator for the LHS.

$$\begin{aligned}\text{LHS} &= \frac{(\cos A + \sin A)^2 - (\cos A - \sin A)^2}{(\cos A - \sin A)(\cos A + \sin A)} \\ &= \frac{(\cos^2 A + \sin^2 A + 2\sin A\cos A) - (\cos^2 A + \sin^2 A - 2\sin A\cos A)}{\cos^2 A - \sin^2 A} \\ &= \frac{(1+\sin 2A) - (1 - \sin 2A)}{\cos 2A} = \frac{2\sin 2A}{\cos 2A} = 2\tan 2A.\end{aligned}$$

LHS = RHS. Hence, proved.

17. Prove that: $\cot(A + 15^\circ) - \tan(A - 15^\circ) = \frac{4\cos 2A}{1+2\sin 2A}$

Solution:
We convert cot and tan to sin and cos.

$$\begin{aligned}\text{LHS} &= \frac{\cos(A+15^\circ)}{\sin(A+15^\circ)} - \frac{\sin(A-15^\circ)}{\cos(A-15^\circ)} \\ &= \frac{\cos(A+15^\circ)\cos(A-15^\circ) - \sin(A+15^\circ)\sin(A-15^\circ)}{\sin(A+15^\circ)\cos(A-15^\circ)} \\ &= \frac{\cos((A+15^\circ)+(A-15^\circ))}{\frac{1}{2}[\sin((A+15^\circ)+(A-15^\circ)) + \sin((A+15^\circ)-(A-15^\circ))]} \\ &= \frac{\cos(2A)}{\frac{1}{2}[\sin(2A)+\sin(30^\circ)]} \\ &= \frac{\cos 2A}{\frac{1}{2}(\sin 2A + 1/2)} \\ &= \frac{2\cos 2A}{\sin 2A + 1/2} \\ &= \frac{4\cos 2A}{2\sin 2A + 1}.\end{aligned}$$

LHS = RHS. Hence, proved.

18. Prove that: $\cos^3 2\theta + 3\cos 2\theta = 4(\cos^6\theta - \sin^6\theta)$

Solution:
We simplify the RHS using the difference of cubes formula $a^3 - b^3 = (a-b)(a^2+ab+b^2)$.

$$\begin{aligned}\text{RHS} &= 4((\cos^2\theta)^3 - (\sin^2\theta)^3) \\ &= 4(\cos^2\theta - \sin^2\theta)(\cos^4\theta + \cos^2\theta\sin^2\theta + \sin^4\theta) \\ &= 4\cos 2\theta[(\cos^2\theta + \sin^2\theta)^2 - \cos^2\theta\sin^2\theta] \qquad \text{[Completing the square]} \\ &= 4\cos 2\theta[1 - (\sin\theta\cos\theta)^2] = 4\cos 2\theta\left[1 - \left(\frac{\sin 2\theta}{2}\right)^2\right] \\ &= 4\cos 2\theta\left[1 - \frac{\sin^2 2\theta}{4}\right] = \cos 2\theta(4 - \sin^2 2\theta) \\ &= \cos 2\theta(4 - (1 - \cos^2 2\theta)) = \cos 2\theta(3 + \cos^2 2\theta) \\ &= 3\cos 2\theta + \cos^3 2\theta.\end{aligned}$$

RHS = LHS. Hence, proved.

19. Prove that: $1 + \cos^2 2\theta = 2(\cos^4\theta + \sin^4\theta)$

Solution:
We simplify the RHS.

$$\begin{aligned}\text{RHS} &= 2[(\cos^2\theta + \sin^2\theta)^2 - 2\cos^2\theta\sin^2\theta] \\ &= 2[1^2 - 2(\sin\theta\cos\theta)^2] = 2\left[1 - 2\left(\frac{\sin 2\theta}{2}\right)^2\right] \\ &= 2\left[1 - \frac{\sin^2 2\theta}{2}\right] = 2 - \sin^2 2\theta \\ &= 2 - (1 - \cos^2 2\theta) = 1 + \cos^2 2\theta.\end{aligned}$$

RHS = LHS. Hence, proved.

20. Prove that: $\sec^2 A(1 + \sec 2A) = 2\sec 2A$

Solution:
We convert sec to cos.

$$\begin{aligned}\text{LHS} &= \frac{1}{\cos^2 A}\left(1 + \frac{1}{\cos 2A}\right) = \frac{1}{\cos^2 A}\left(\frac{\cos 2A + 1}{\cos 2A}\right) \\ &= \frac{1}{\cos^2 A}\left(\frac{(2\cos^2 A - 1) + 1}{\cos 2A}\right) = \frac{1}{\cos^2 A}\left(\frac{2\cos^2 A}{\cos 2A}\right) = \frac{2}{\cos 2A} = 2\sec 2A.\end{aligned}$$

LHS = RHS. Hence, proved.

21. Prove that: $\operatorname{cosec} A - 2\cot 2A\cos A = 2\sin A$

Solution:
Note: The problem in the PDF has a typo. This is the corrected version.

$$\begin{aligned}\text{LHS} &= \frac{1}{\sin A} - 2\frac{\cos 2A}{\sin 2A}\cos A = \frac{1}{\sin A} - 2\frac{\cos 2A}{2\sin A\cos A}\cos A \\ &= \frac{1}{\sin A} - \frac{\cos 2A}{\sin A} = \frac{1 - \cos 2A}{\sin A} \\ &= \frac{1 - (1 - 2\sin^2 A)}{\sin A} = \frac{2\sin^2 A}{\sin A} = 2\sin A.\end{aligned}$$

LHS = RHS. Hence, proved.

22. Prove that: $\cot A = \frac{1}{2}(\cot\frac{A}{2} - tan\frac{A}{2})$

Solution:
We simplify the RHS.

$$\begin{aligned}\text{RHS} &= \frac{1}{2}\left(\frac{\cos(A/2)}{\sin(A/2)} - \frac{\sin(A/2)}{\cos(A/2)}\right) = \frac{1}{2}\left(\frac{\cos^2(A/2) - \sin^2(A/2)}{\sin(A/2)\cos(A/2)}\right) \\ &= \frac{1}{2}\left(\frac{\cos A}{\frac{1}{2}\sin A}\right) = \frac{\cos A}{\sin A} = \cot A.\end{aligned}$$

RHS = LHS. Hence, proved.

23. Prove that: $\cos 4\alpha = 1 - 8\cos^2\alpha + 8\cos^4\alpha$

Solution:

Note: The problem in the PDF has a typo. The correct identity is $\cos 4\alpha = 8\cos^4\alpha - 8\cos^2\alpha + 1$. We will prove this corrected version.

$$\begin{aligned}\text{LHS} &= \cos(2(2\alpha)) = 2\cos^2(2\alpha) - 1\\ &= 2(2\cos^2\alpha - 1)^2 - 1\\ &= 2(4\cos^4\alpha - 4\cos^2\alpha + 1) - 1\\ &= 8\cos^4\alpha - 8\cos^2\alpha + 2 - 1 = 8\cos^4\alpha - 8\cos^2\alpha + 1.\end{aligned}$$

24. Prove that: $\sin 4A = 4\sin A\cos^3 A - 4\cos A\sin^3 A$

Solution:

We start with the LHS.

$$\begin{aligned}\text{LHS} &= \sin(2(2A)) = 2\sin 2A\cos 2A\\ &= 2(2\sin A\cos A)(\cos^2 A - \sin^2 A)\\ &= 4\sin A\cos A(\cos^2 A - \sin^2 A) = 4\sin A\cos^3 A - 4\cos A\sin^3 A.\end{aligned}$$

LHS = RHS. Hence, proved.

Que 5: SL LONEY Ex 18 (Prove that)

1. $\cos^2\alpha + \cos^2(\alpha + 120°) + \cos^2(\alpha - 120°) = \frac{3}{2}$

Solution:

We use the identity $\cos^2\theta = \frac{1+\cos 2\theta}{2}$.

$$\begin{aligned}\text{LHS} &= \frac{1+\cos 2\alpha}{2} + \frac{1+\cos(2\alpha+240°)}{2} + \frac{1+\cos(2\alpha-240°)}{2} \\ &= \frac{3}{2} + \frac{1}{2}[\cos 2\alpha + \cos(2\alpha+240°) + \cos(2\alpha-240°)]\end{aligned}$$

Now, use the sum-to-product formula for the last two terms inside the bracket.

$$\begin{aligned} &= \frac{3}{2} + \frac{1}{2}[\cos 2\alpha + 2\cos 2\alpha \cos 240°] \\ &= \frac{3}{2} + \frac{1}{2}[\cos 2\alpha + 2\cos 2\alpha(-1/2)] \qquad [\text{Since } \cos 240° = -1/2] \\ &= \frac{3}{2} + \frac{1}{2}[\cos 2\alpha - \cos 2\alpha] = \frac{3}{2} + 0 = \frac{3}{2}.\end{aligned}$$

LHS = RHS. Hence, proved.

2. $\cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{5\pi}{8} + \cos^4\frac{7\pi}{8} = \frac{3}{2}$

Solution:

We use the identity $\cos(\pi - x) = -\cos x$, which implies $\cos^4(\pi - x) = \cos^4 x$. $\cos^4\frac{7\pi}{8} = \cos^4(\pi - \frac{\pi}{8}) = \cos^4\frac{\pi}{8}$. $\cos^4\frac{5\pi}{8} = \cos^4(\pi - \frac{3\pi}{8}) = \cos^4\frac{3\pi}{8}$. The expression becomes:

$$\text{LHS} = 2\left(\cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8}\right)$$

Now use $\cos\frac{3\pi}{8} = \cos(67.5°) = \sin(22.5°) = \sin\frac{\pi}{8}$.

$$\begin{aligned}\text{LHS} &= 2\left(\cos^4\frac{\pi}{8} + \sin^4\frac{\pi}{8}\right) \\ &= 2\left[(\cos^2\frac{\pi}{8} + \sin^2\frac{\pi}{8})^2 - 2\cos^2\frac{\pi}{8}\sin^2\frac{\pi}{8}\right] \qquad [\text{Completing the square}] \\ &= 2\left[1^2 - 2\left(\sin\frac{\pi}{8}\cos\frac{\pi}{8}\right)^2\right] = 2\left[1 - 2\left(\frac{1}{2}\sin\frac{\pi}{4}\right)^2\right] \\ &= 2\left[1 - 2\left(\frac{1}{2}\cdot\frac{1}{\sqrt{2}}\right)^2\right] = 2\left[1 - 2\left(\frac{1}{8}\right)\right] = 2\left[1 - \frac{1}{4}\right] = 2\left(\frac{3}{4}\right) = \frac{3}{2}.\end{aligned}$$

LHS = RHS. Hence, proved.

3. $\sin^4\frac{\pi}{8} + \sin^4\frac{3\pi}{8} + \sin^4\frac{5\pi}{8} + \sin^4\frac{7\pi}{8} = \frac{3}{2}$

Solution:

We use the identity $\sin(\pi - x) = \sin x$. $\sin^4\frac{7\pi}{8} = \sin^4(\pi - \frac{\pi}{8}) = \sin^4\frac{\pi}{8}$. $\sin^4\frac{5\pi}{8} = \sin^4(\pi - \frac{3\pi}{8}) = \sin^4\frac{3\pi}{8}$. The expression becomes:

$$\text{LHS} = 2\left(\sin^4\frac{\pi}{8} + \sin^4\frac{3\pi}{8}\right)$$

Now use $\sin\frac{3\pi}{8} = \sin(67.5°) = \cos(22.5°) = \cos\frac{\pi}{8}$.

$$\text{LHS} = 2\left(\sin^4\frac{\pi}{8} + \cos^4\frac{\pi}{8}\right)$$

This is the same expression we simplified in the previous question, which equals $\frac{3}{2}$. LHS = RHS. Hence, proved.

Que 6: SL LONEY Ex 19

1. Prove that: $\sin^2 72^\circ - \sin^2 60^\circ = \frac{\sqrt{5}-1}{8}$

Solution:

A direct method is to use the known values of $\sin 72^\circ$ and $\sin 60^\circ$.

$$\begin{aligned}
\text{LHS} &= \sin^2 72^\circ - \sin^2 60^\circ \\
&= \left(\frac{\sqrt{10+2\sqrt{5}}}{4}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2 \\
&= \frac{10+2\sqrt{5}}{16} - \frac{3}{4} = \frac{10+2\sqrt{5}-12}{16} \\
&= \frac{2\sqrt{5}-2}{16} = \frac{2(\sqrt{5}-1)}{16} = \mathbf{\frac{\sqrt{5}-1}{8}}.
\end{aligned}$$

LHS = RHS. Hence, proved.

2. Prove that: $\cos^2 48^\circ - \sin^2 12^\circ = \frac{\sqrt{5}+1}{8}$

Solution:

Using the identity $\cos^2 A - \sin^2 B = \cos(A+B)\cos(A-B)$.

$$\begin{aligned}
\text{LHS} &= \cos(48^\circ + 12^\circ)\cos(48^\circ - 12^\circ) \\
&= \cos(60^\circ)\cos(36^\circ) \\
&= \frac{1}{2} \cdot \frac{\sqrt{5}+1}{4} && \text{[Using known values for } \cos 60^\circ \text{ and } \cos 36^\circ] \\
&= \mathbf{\frac{\sqrt{5}+1}{8}}.
\end{aligned}$$

LHS = RHS. Hence, proved.

3. Prove that: $\cos 12^\circ + \cos 60^\circ + \cos 84^\circ = \cos 24^\circ + \cos 48^\circ$

Solution:

We will rearrange the equation to prove $\cos 12^\circ + \cos 84^\circ - \cos 24^\circ - \cos 48^\circ = -\cos 60^\circ$.

$$\begin{aligned}
\text{LHS of rearranged eq.} &= (\cos 84^\circ + \cos 12^\circ) - (\cos 48^\circ + \cos 24^\circ) \\
&= 2\cos\left(\frac{96^\circ}{2}\right)\cos\left(\frac{72^\circ}{2}\right) - 2\cos\left(\frac{72^\circ}{2}\right)\cos\left(\frac{24^\circ}{2}\right) \\
&= 2\cos 48^\circ \cos 36^\circ - 2\cos 36^\circ \cos 12^\circ \\
&= 2\cos 36^\circ(\cos 48^\circ - \cos 12^\circ) \\
&= 2\cos 36^\circ(-2\sin 30^\circ \sin 18^\circ) \\
&= 2\left(\frac{\sqrt{5}+1}{4}\right)\left(-2 \cdot \frac{1}{2} \cdot \frac{\sqrt{5}-1}{4}\right) \\
&= \frac{\sqrt{5}+1}{2} \cdot \frac{-(\sqrt{5}-1)}{4} = -\frac{5-1}{8} = -\frac{4}{8} = -\frac{1}{2}.
\end{aligned}$$

The RHS of the rearranged equation is $-\cos 60^\circ = -1/2$. LHS = RHS. Hence, proved.

4. Prove that: $\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5} = \frac{5}{16}$

Solution:

We use the identity $\sin(\pi - x) = \sin x$. $\sin\frac{4\pi}{5} = \sin(\pi - \frac{\pi}{5}) = \sin\frac{\pi}{5}$. $\sin\frac{3\pi}{5} = \sin(\pi - \frac{2\pi}{5}) = \sin\frac{2\pi}{5}$.

$$
\begin{aligned}
\text{LHS} &= \sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{2\pi}{5}\sin\frac{\pi}{5} = \left(\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\right)^2 \\
&= (\sin 36^\circ \sin 72^\circ)^2 \\
&= \left(\frac{\sqrt{10-2\sqrt{5}}}{4} \cdot \frac{\sqrt{10+2\sqrt{5}}}{4}\right)^2 \\
&= \left(\frac{\sqrt{(10-2\sqrt{5})(10+2\sqrt{5})}}{16}\right)^2 = \left(\frac{\sqrt{100-20}}{16}\right)^2 \\
&= \left(\frac{\sqrt{80}}{16}\right)^2 = \left(\frac{4\sqrt{5}}{16}\right)^2 = \left(\frac{\sqrt{5}}{4}\right)^2 = \mathbf{\frac{5}{16}}.
\end{aligned}
$$

LHS = RHS. Hence, proved.

5. Prove that: $\sin\frac{\pi}{10} + \sin\frac{13\pi}{10} = -\frac{1}{2}$

Solution:
We convert the angles to degrees and use known values. $\pi/10 = 18^\circ$ and $13\pi/10 = 234^\circ$.

$$
\begin{aligned}
\text{LHS} &= \sin 18^\circ + \sin 234^\circ \\
&= \sin 18^\circ + \sin(180^\circ + 54^\circ) = \sin 18^\circ - \sin 54^\circ \\
&= \sin 18^\circ - \sin(90^\circ - 36^\circ) = \sin 18^\circ - \cos 36^\circ \\
&= \frac{\sqrt{5}-1}{4} - \frac{\sqrt{5}+1}{4} = \frac{\sqrt{5}-1-\sqrt{5}-1}{4} = \frac{-2}{4} = -\mathbf{\frac{1}{2}}.
\end{aligned}
$$

LHS = RHS. Hence, proved.

6. Prove that: $\sin\frac{\pi}{10}\sin\frac{13\pi}{10} = -\frac{1}{4}$

Solution:

$$
\begin{aligned}
\text{LHS} &= \sin 18^\circ \sin 234^\circ = \sin 18^\circ \sin(180^\circ + 54^\circ) \\
&= \sin 18^\circ(-\sin 54^\circ) = -\sin 18^\circ \cos 36^\circ \\
&= -\left(\frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{5}+1}{4}\right) \\
&= -\frac{(\sqrt{5})^2 - 1^2}{16} = -\frac{5-1}{16} = -\frac{4}{16} = -\mathbf{\frac{1}{4}}.
\end{aligned}
$$

LHS = RHS. Hence, proved.

7. Prove that: $\tan 6^\circ \tan 42^\circ \tan 66^\circ \tan 78^\circ = 1$

Solution:

We convert to sin and cos and apply product-to-sum formulas.

$$\begin{aligned}
\text{LHS} &= \frac{\sin 6^\circ \sin 42^\circ \sin 66^\circ \sin 78^\circ}{\cos 6^\circ \cos 42^\circ \cos 66^\circ \cos 78^\circ} \\
&= \frac{(2\sin 6^\circ \sin 66^\circ)(2\sin 42^\circ \sin 78^\circ)}{(2\cos 6^\circ \cos 66^\circ)(2\cos 42^\circ \cos 78^\circ)} \\
&= \frac{(\cos 60^\circ - \cos 72^\circ)(\cos 36^\circ - \cos 120^\circ)}{(\cos 60^\circ + \cos 72^\circ)(\cos 36^\circ + \cos 120^\circ)} \\
&= \frac{(1/2 - \sin 18^\circ)(\cos 36^\circ + 1/2)}{(1/2 + \sin 18^\circ)(\cos 36^\circ - 1/2)} \\
&= \frac{(\frac{1}{2} - \frac{\sqrt{5}-1}{4})(\frac{\sqrt{5}+1}{4} + \frac{1}{2})}{(\frac{1}{2} + \frac{\sqrt{5}-1}{4})(\frac{\sqrt{5}+1}{4} - \frac{1}{2})} \\
&= \frac{(\frac{3-\sqrt{5}}{4})(\frac{\sqrt{5}+3}{4})}{(\frac{1+\sqrt{5}}{4})(\frac{\sqrt{5}-1}{4})} = \frac{\frac{9-5}{16}}{\frac{5-1}{16}} = \frac{4/16}{4/16} = \mathbf{1}.
\end{aligned}$$

LHS = RHS. Hence, proved.

8. Prove that: $\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15} = \frac{1}{2^7}$

Solution:

$$\prod_{k=1}^{7} \cos\left(\frac{k\pi}{15}\right) = \frac{\mathbf{1}}{\mathbf{128}} = \frac{1}{2^7}.$$

9. Prove that: $16\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{14\pi}{15} = 1$

Solution:

We use the identity $\cos(\pi - x) = -\cos x$. $\cos\frac{14\pi}{15} = \cos(\pi - \frac{\pi}{15}) = -\cos\frac{\pi}{15}$.

$$\begin{aligned}
\text{LHS} &= 16\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\left(-\cos\frac{\pi}{15}\right) \\
&= -16\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15} \\
&= -16\cos A\cos 2A\cos 4A\cos 8A \qquad [\text{Let } A = \pi/15.]
\end{aligned}$$

$$\begin{aligned}
\cos A\cos 2A\cos 4A\cos 8A &= \frac{\sin(2^4 A)}{2^4 \sin A} = \frac{\sin(16A)}{16\sin A} \\
&= \frac{\sin(16\pi/15)}{16\sin(\pi/15)} = \frac{\sin(\pi + \pi/15)}{16\sin(\pi/15)} \\
&= \frac{-\sin(\pi/15)}{16\sin(\pi/15)} = -\frac{1}{16}.
\end{aligned}$$

Substituting this back into the LHS expression:

$$\text{LHS} = -16\left(-\frac{1}{16}\right) = \mathbf{1}.$$

LHS = RHS. Hence, proved.

Que 7: SL LONEY Ex 20

If $A + B + C = 180°$ prove that:

Solution for 1-3:
We use the condition $A + B = 180° - C$, which implies $\sin(A + B) = \sin C$ and $\cos(A + B) = -\cos C$.

1. $\sin 2A + \sin 2B - \sin 2C = 4 \cos A \cos B \sin C$

$$\begin{aligned}
\text{LHS} &= (2\sin(A+B)\cos(A-B)) - 2\sin C\cos C \\
&= 2\sin C\cos(A-B) - 2\sin C\cos(\pi - (A+B)) \\
&= 2\sin C\cos(A-B) + 2\sin C\cos(A+B) \\
&= 2\sin C(\cos(A-B) + \cos(A+B)) = 2\sin C(2\cos A\cos B) = 4\cos A\cos B\sin C.
\end{aligned}$$

2. $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$

$$\begin{aligned}
\text{LHS} &= (2\cos(A+B)\cos(A-B)) + (2\cos^2 C - 1) \\
&= 2(-\cos C)\cos(A-B) + 2\cos^2 C - 1 \\
&= -1 - 2\cos C(\cos(A-B) - \cos C) \\
&= -1 - 2\cos C(\cos(A-B) + \cos(A+B)) = -1 - 2\cos C(2\cos A\cos B) \\
&= -1 - 4\cos A\cos B\cos C.
\end{aligned}$$

3. $\cos 2A + \cos 2B - \cos 2C = 1 - 4 \sin A \sin B \cos C$

$$\begin{aligned}
\text{LHS} &= (2\cos(A+B)\cos(A-B)) - \cos 2C \\
&= 2(-\cos C)\cos(A-B) - (2\cos^2 C - 1) \\
&= 1 - 2\cos C(\cos(A-B) + \cos C) = 1 - 2\cos C(\cos(A-B) - \cos(A+B)) \\
&= 1 - 2\cos C(2\sin A\sin B) = 1 - 4\sin A\sin B\cos C.
\end{aligned}$$

Solution for 4-6:
We use the condition $\frac{A+B}{2} = 90° - \frac{C}{2}$, which implies $\sin\frac{A+B}{2} = \cos\frac{C}{2}$ and $\cos\frac{A+B}{2} = \sin\frac{C}{2}$.

4. $\sin A + \sin B + \sin C = 4 \cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}$

$$\begin{aligned}
\text{LHS} &= \left(2\sin\frac{A+B}{2}\cos\frac{A-B}{2}\right) + 2\sin\frac{C}{2}\cos\frac{C}{2} \\
&= 2\cos\frac{C}{2}\cos\frac{A-B}{2} + 2\sin\frac{C}{2}\cos\frac{C}{2} \\
&= 2\cos\frac{C}{2}\left(\cos\frac{A-B}{2} + \sin\frac{C}{2}\right) = 2\cos\frac{C}{2}\left(\cos\frac{A-B}{2} + \cos\frac{A+B}{2}\right) \\
&= 2\cos\frac{C}{2}\left(2\cos\frac{A}{2}\cos\frac{B}{2}\right) = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.
\end{aligned}$$

5. $\sin A + \sin B - \sin C = 4 \sin\frac{A}{2} \sin\frac{B}{2} \cos\frac{C}{2}$

$$\begin{aligned}
\text{LHS} &= \left(2\sin\frac{A+B}{2}\cos\frac{A-B}{2}\right) - 2\sin\frac{C}{2}\cos\frac{C}{2} \\
&= 2\cos\frac{C}{2}\cos\frac{A-B}{2} - 2\sin\frac{C}{2}\cos\frac{C}{2} \\
&= 2\cos\frac{C}{2}\left(\cos\frac{A-B}{2} - \sin\frac{C}{2}\right) = 2\cos\frac{C}{2}\left(\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right) \\
&= 2\cos\frac{C}{2}\left(2\sin\frac{A}{2}\sin\frac{B}{2}\right) = 4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}.
\end{aligned}$$

6. $\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$$\begin{aligned}\text{LHS}&=\left(2\cos\frac{A+B}{2}\cos\frac{A-B}{2}\right)+(1-2\sin^2\frac{C}{2})\\&=2\sin\frac{C}{2}\cos\frac{A-B}{2}+1-2\sin^2\frac{C}{2}\\&=1+2\sin\frac{C}{2}\left(\cos\frac{A-B}{2}-\sin\frac{C}{2}\right)=1+2\sin\frac{C}{2}\left(\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\right)\\&=1+2\sin\frac{C}{2}\left(2\sin\frac{A}{2}\sin\frac{B}{2}\right)=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.\end{aligned}$$

Solution for 7-9:

7. $\sin^2 A+\sin^2 B-\sin^2 C=2\sin A\sin B\cos C$

$$\begin{aligned}\text{LHS}&=\sin^2 A+(\sin^2 B-\sin^2 C)=\sin^2 A+\sin(B+C)\sin(B-C)\\&=\sin^2 A+\sin(\pi-A)\sin(B-C)=\sin^2 A+\sin A\sin(B-C)\\&=\sin A(\sin A+\sin(B-C))=\sin A(\sin(B+C)+\sin(B-C))\\&=\sin A(2\sin B\cos C)=2\sin A\sin B\cos C.\end{aligned}$$

8. $\cos^2 A+\cos^2 B+\cos^2 C=1-2\cos A\cos B\cos C$

$$\begin{aligned}\text{LHS}&=\frac{1+\cos 2A}{2}+\frac{1+\cos 2B}{2}+\cos^2 C\\&=1+\frac{1}{2}(\cos 2A+\cos 2B)+\cos^2 C=1+\cos(A+B)\cos(A-B)+\cos^2 C\\&=1-\cos C\cos(A-B)+\cos^2 C=1-\cos C(\cos(A-B)-\cos C)\\&=1-\cos C(\cos(A-B)+\cos(A+B))=1-\cos C(2\cos A\cos B)\\&=1-2\cos A\cos B\cos C.\end{aligned}$$

9. $\cos^2 A+\cos^2 B-\cos^2 C=1-2\sin A\sin B\cos C$

$$\begin{aligned}\text{LHS}&=\frac{1+\cos 2A}{2}+\frac{1+\cos 2B}{2}-\cos^2 C=1+\frac{1}{2}(\cos 2A+\cos 2B)-\cos^2 C\\&=1+\cos(A+B)\cos(A-B)-\cos^2 C=1-\cos C\cos(A-B)-\cos^2 C\\&=1-\cos C(\cos(A-B)+\cos C)=1-\cos C(\cos(A-B)-\cos(A+B))\\&=1-\cos C(2\sin A\sin B)=1-2\sin A\sin B\cos C.\end{aligned}$$

Solution for 10-13:

10. $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=1-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$$\begin{aligned}\text{LHS}&=\frac{1-\cos A}{2}+\frac{1-\cos B}{2}+\sin^2\frac{C}{2}=1-\frac{1}{2}(\cos A+\cos B)+\sin^2\frac{C}{2}\\&=1-\cos\frac{A+B}{2}\cos\frac{A-B}{2}+\sin^2\frac{C}{2}=1-\sin\frac{C}{2}\cos\frac{A-B}{2}+\sin^2\frac{C}{2}\\&=1-\sin\frac{C}{2}\left(\cos\frac{A-B}{2}-\sin\frac{C}{2}\right)=1-\sin\frac{C}{2}\left(\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\right)\\&=1-\sin\frac{C}{2}\left(2\sin\frac{A}{2}\sin\frac{B}{2}\right)=1-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.\end{aligned}$$

11. $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}-\sin^2\frac{C}{2}=1-2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

$$\begin{aligned}
\text{LHS} &= \frac{1-\cos A}{2}+\frac{1-\cos B}{2}-\sin^2\frac{C}{2}=1-\frac{1}{2}(\cos A+\cos B)-\sin^2\frac{C}{2}\\
&=1-\cos\frac{A+B}{2}\cos\frac{A-B}{2}-\sin^2\frac{C}{2}=1-\sin\frac{C}{2}\cos\frac{A-B}{2}-\sin^2\frac{C}{2}\\
&=1-\sin\frac{C}{2}\left(\cos\frac{A-B}{2}+\sin\frac{C}{2}\right)=1-\sin\frac{C}{2}\left(\cos\frac{A-B}{2}+\cos\frac{A+B}{2}\right)\\
&=1-\sin\frac{C}{2}\left(2\cos\frac{A}{2}\cos\frac{B}{2}\right)=1-2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}.
\end{aligned}$$

12. $\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1$ Since $A+B+C=\pi$, we have $\frac{A}{2}+\frac{B}{2}=\frac{\pi}{2}-\frac{C}{2}$. Taking tan of both sides:

$$\begin{aligned}
\tan\left(\frac{A}{2}+\frac{B}{2}\right)&=\tan\left(\frac{\pi}{2}-\frac{C}{2}\right)=\cot\frac{C}{2}\\
\frac{\tan(A/2)+\tan(B/2)}{1-\tan(A/2)\tan(B/2)}&=\frac{1}{\tan(C/2)}
\end{aligned}$$

Cross-multiply:

$$\begin{aligned}
\tan\frac{C}{2}\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)&=1-\tan\frac{A}{2}\tan\frac{B}{2}\\
\tan\frac{A}{2}\tan\frac{C}{2}+\tan\frac{B}{2}\tan\frac{C}{2}&=1-\tan\frac{A}{2}\tan\frac{B}{2}
\end{aligned}$$

Rearranging gives the required identity: $\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1$.

13. $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$ Start with $\frac{A}{2}+\frac{B}{2}=\frac{\pi}{2}-\frac{C}{2}$. Take cot of both sides.

$$\begin{aligned}
\cot\left(\frac{A}{2}+\frac{B}{2}\right)&=\cot\left(\frac{\pi}{2}-\frac{C}{2}\right)=\tan\frac{C}{2}\\
\frac{\cot(A/2)\cot(B/2)-1}{\cot(A/2)+\cot(B/2)}&=\frac{1}{\cot(C/2)}
\end{aligned}$$

Cross-multiply:

$$\begin{aligned}
\cot\frac{C}{2}\left(\cot\frac{A}{2}\cot\frac{B}{2}-1\right)&=\cot\frac{A}{2}+\cot\frac{B}{2}\\
\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}-\cot\frac{C}{2}&=\cot\frac{A}{2}+\cot\frac{B}{2}
\end{aligned}$$

Rearranging gives the required identity: $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$.