Hi everyone,

The real learning in mathematics happens when you actively engage with a problem, explore different methods, and work through challenges. Therefore, we strongly encourage you to use this solution key responsibly.

Please attempt all the problems on your own first, giving them your best and most honest effort. These solutions are to help you get unstuck on a problem after you have already tried your best.

Your effort and dedication are the true keys to success.

JEE Repeater CT-04: 20 July 2025

Topic: Sequence and Series

Sub: Mathematics **JEE Main CT-04** Prof. Chetan Sir

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51. In a cricket tournament 16 school teams participated. A sum of Rs. 8000 is to be awarded among themselves as prize money. If the last placed team is awarded Rs. 275 in prize money and the award increases by the same amount for successive finishing places, amount will the first place team received is

(1) Rs.720 (2) Rs. 725 (3) Rs.735 (4) Rs.780

Solution:

Concept Used:

- Sum of an Arithmetic Progression (AP): $S_n = \frac{n}{2}[2a + (n-1)d]$.
- The nth term of an AP: $a_n = a + (n-1)d$.

Hint:

- The prize money forms an AP. Let the prize for the last team (16th place) be the first term of the AP.
- Use the sum formula to find the common difference.
- Calculate the 16th term of this AP, which corresponds to the prize for the first-place team.

Let the prize money form an AP. It's convenient to define the first term 'a' as the prize for the last place
Number of teams, $n = 16$.
Prize for the last team (16th place), $a = 275$.
Total sum of prizes, $S_{16} = 8000$.
Let 'd' be the common difference by which the prize money increases.

Using the sum formula for an AP:
$S_n = \frac{n}{2}[2a + (n-1)d]$.
$8000 = \frac{16}{2}[2(275) + (16-1)d]$.
$8000 = 8[550 + 15d]$.
$1000 = 550 + 15d$.
$15d = 1000 - 550 = 450$.
$d = \frac{450}{15} = 30$.

The first place team receives the 16th term of this AP.
$a_{16} = a + (16-1)d = a + 15d$.
$a_{16} = 275 + 15(30) = 275 + 450 = 725$.
The first place team will receive Rs. 725.

The correct option is **(2)**.

52. Let S_1 be the sum of first 2n terms of an arithmetic progression. Let S_2 be the sum of first 4n terms of the same arithmetic progression. If $(S_2 - S_1)$ is 1000, then the sum of the first 6n terms of the arithmetic progression is $a \times 10^3$ then a is:

(1) 1 (2) 2 (3) 3 (4) 4

Solution:

Concept Used:

- Sum of an AP.
- The difference $S_{4n} - S_{2n}$ is the sum of terms from the $(2n+1)^{th}$ to the $4n^{th}$ term.

Hint:

- Express S_{2n}, S_{4n} and S_{6n} using the formula $S_k = \frac{k}{2}[2a + (k-1)d]$.
- Use the given condition $S_{4n} - S_{2n} = 1000$ to find an expression relating n, a, d.
- Notice the structural similarity between the expression for $S_{4n} - S_{2n}$ and S_{6n}.

$$S_{2n} = \frac{2n}{2}[2a + (2n-1)d] = n[2a + (2n-1)d].$$
$$S_{4n} = \frac{4n}{2}[2a + (4n-1)d] = 2n[2a + (4n-1)d].$$

Given $S_{4n} - S_{2n} = 1000$.

$$2n[2a + (4n-1)d] - n[2a + (2n-1)d] = 1000.$$
$$n[2(2a + 4nd - d) - (2a + 2nd - d)] = 1000.$$
$$n[4a + 8nd - 2d - 2a - 2nd + d] = 1000.$$
$$n[2a + 6nd - d] = n[2a + (6n-1)d] = 1000.$$

We need to find the sum of the first 6n terms, S_{6n}.

$$S_{6n} = \frac{6n}{2}[2a + (6n-1)d].$$
$$S_{6n} = 3n[2a + (6n-1)d].$$

From our previous result, we know $n[2a + (6n-1)d] = 1000$.

$S_{6n} = 3 \times (1000) = 3000$.

We are given $S_{6n} = a \times 10^3$.

$a \times 1000 = 3000 \implies a = 3$.

The correct option is **(3)**.

53. $S_1 = 3, 7, 11, 15, ...$ upto 125 terms and $S_2 = 4, 7, 10, 13, 16, ...$ upto 125 terms, then how many terms are there in S_1 that are there in S_2?

(1) 29 (2) 30 (3) 31 (4) 28

Solution:

Concept Used:

- Properties of common terms of two APs.
- The common terms of two APs also form an AP.
- The common difference of the new AP is the LCM of the original common differences.

Hint:

- Identify the first common term.
- Find the common difference of the AP of common terms.
- Determine the last possible value for a common term by finding the last term of both sequences.
- Use the formula for the nth term to find the number of common terms.

For sequence S_1 : First term $a_1 = 3$, common difference $d_1 = 4$.
For sequence S_2 : First term $a_2 = 4$, common difference $d_2 = 3$.

The first common term is clearly 7.
The common difference of the AP of common terms is $d = \text{LCM}(d_1, d_2) = \text{LCM}(4, 3) = 12$.
So, the sequence of common terms is an AP with first term $a_c = 7$ and common difference $d_c = 12$.
The common terms are $7, 19, 31, \dots$

Now, we find the last term of each sequence to set a boundary.
Last term of $S_1 : T_{125}^{(1)} = 3 + (125 - 1) \times 4 = 3 + 124 \times 4 = 3 + 496 = 499$.
Last term of $S_2 : T_{125}^{(2)} = 4 + (125 - 1) \times 3 = 4 + 124 \times 3 = 4 + 372 = 376$.
Any common term must be less than or equal to $\min(499, 376) = 376$.

Let the nth term of the common AP be $T_n^{(c)}$.We need $T_n^{(c)} \leq 376$.
$a_c + (n - 1)d_c \leq 376$.
$7 + (n - 1)12 \leq 376$.
$(n - 1)12 \leq 369$.
$n - 1 \leq \frac{369}{12} = 30.75$.
$n \leq 31.75$.
Since n must be an integer, the maximum value of n is 31.
There are 31 common terms.

The correct option is **(3)**.

54. If $t_n = \frac{1}{4}(n+2)(n+3)$ for $n = 1, 2, 3, ...$, then $\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + ... + \frac{1}{t_{2003}} =$

(1) $\frac{4006}{3006}$ (2) $\frac{4003}{3007}$ (3) $\frac{4006}{3008}$ (4) $\frac{4006}{3009}$

Solution:

Concept Used:

- Method of difference for summation (Telescoping series).
- Partial fraction decomposition: $\frac{1}{(x+a)(x+b)} = \frac{1}{b-a}\left(\frac{1}{x+a} - \frac{1}{x+b}\right)$.

Given $t_n = \dfrac{1}{4}(n+2)(n+3)$.

The term in the sum is $\dfrac{1}{t_n} = \dfrac{4}{(n+2)(n+3)}$.

Using partial fractions, we can write:

$$\frac{4}{(n+2)(n+3)} = 4\left(\frac{1}{n+2} - \frac{1}{n+3}\right).$$

The required sum S is:

$$S = \sum_{n=1}^{2003} 4\left(\frac{1}{n+2} - \frac{1}{n+3}\right).$$

$$S = 4\sum_{n=1}^{2003} \left(\frac{1}{n+2} - \frac{1}{n+3}\right).$$

This is a telescoping series. Let's write out the terms:

$$n = 1: \quad 4\left(\frac{1}{3} - \frac{1}{4}\right)$$

$$n = 2: \quad 4\left(\frac{1}{4} - \frac{1}{5}\right)$$

$$n = 3: \quad 4\left(\frac{1}{5} - \frac{1}{6}\right)$$

$$\vdots$$

$$n = 2003: \quad 4\left(\frac{1}{2005} - \frac{1}{2006}\right)$$

Adding these terms, all intermediate terms cancel out, leaving only the first and the last.

$$S = 4\left(\frac{1}{3} - \frac{1}{2006}\right).$$

$$S = 4\left(\frac{2006-3}{3\times 2006}\right) = 4\left(\frac{2003}{6018}\right).$$

$$S = \frac{8012}{6018} = \frac{4006}{3009}.$$

The correct option is **(4)**.

55. If $(10)^9 + 2(11)^1(10)^8 + 3(11)^2(10)^7 + ...10(11)^9 = k(10)^9$ then k is equal to :

(1) 110　　(2) $\frac{121}{10}$　　(3) $\frac{441}{100}$　　(4) 100

Solution:

Concept Used:

- Sum of an Arithmetico-Geometric Progression (AGP).

Hint:

- Let the given sum be S. Divide the entire equation by 10^9 to simplify the expression.
- Recognize the resulting series as an AGP.
- Use the standard method for summing an AGP: Let the sum be P, calculate P - rP, where r is the common ratio.

Let the given sum be $S = (10)^9 + 2(11)^1(10)^8 + 3(11)^2(10)^7 + \cdots + 10(11)^9$.
Divide the equation $S = k(10)^9$ by 10^9 :

$$k = \frac{S}{10^9} = 1 + 2\frac{11}{10} + 3\left(\frac{11}{10}\right)^2 + \cdots + 10\left(\frac{11}{10}\right)^9.$$

This is an AGP. Let $x = \dfrac{11}{10}$. The series for k is:

$k = 1 + 2x + 3x^2 + \cdots + 10x^9. \quad \cdots(1)$

$xk = \quad x + 2x^2 + \cdots + 9x^9 + 10x^{10}. \quad \cdots(2)$

Subtract (2) from (1):

$k(1-x) = (1 + x + x^2 + \cdots + x^9) - 10x^{10}.$

The term in the parenthesis is a GP with 10 terms.

$$k(1-x) = \frac{x^{10}-1}{x-1} - 10x^{10}.$$

Substitute back $x = 11/10$. Then $1 - x = -1/10$ and $x - 1 = 1/10$.

$$k\left(-\frac{1}{10}\right) = \frac{(11/10)^{10}-1}{1/10} - 10\left(\frac{11}{10}\right)^{10}.$$

$$-\frac{k}{10} = 10\left(\left(\frac{11}{10}\right)^{10} - 1\right) - 10\left(\frac{11}{10}\right)^{10}.$$

$$-\frac{k}{10} = 10\left(\frac{11}{10}\right)^{10} - 10 - 10\left(\frac{11}{10}\right)^{10}.$$

$$-\frac{k}{10} = -10.$$

$k = 100.$

The correct option is **(4)**.

56. If $a_1, a_2, a_3, ...a_{4001}$ are terms of an AP such that $\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + ... + \frac{1}{a_{4000} a_{4001}} = 10$ and $a_2 + a_{4000} = 50$ then $|a_1 - a_{4001}|$ is equal to

(1) 10 (2) 30 (3) 20 (4) 50

Solution:

Concept Used:

- Method of difference / Telescoping series.
- Properties of an AP.

Hint:

- Express the general term $\frac{1}{a_k a_{k+1}}$ as $\frac{1}{d}\left(\frac{1}{a_k} - \frac{1}{a_{k+1}}\right)$.
- Use the telescoping sum to simplify the first given equation.
- Use the property $a_2 + a_{4000} = a_1 + a_{4001}$.

Let d be the common difference of the AP. The general term of the sum is $\frac{1}{a_k a_{k+1}}$.

$$\frac{1}{a_k a_{k+1}} = \frac{1}{d}\left(\frac{d}{a_k a_{k+1}}\right) = \frac{1}{d}\left(\frac{a_{k+1} - a_k}{a_k a_{k+1}}\right) = \frac{1}{d}\left(\frac{1}{a_k} - \frac{1}{a_{k+1}}\right).$$

The given sum is a telescoping series:

$$S = \sum_{k=1}^{4000} \frac{1}{a_k a_{k+1}} = \frac{1}{d}\sum_{k=1}^{4000}\left(\frac{1}{a_k} - \frac{1}{a_{k+1}}\right).$$

$$S = \frac{1}{d}\left[\left(\frac{1}{a_1} - \frac{1}{a_2}\right) + \left(\frac{1}{a_2} - \frac{1}{a_3}\right) + \cdots + \left(\frac{1}{a_{4000}} - \frac{1}{a_{4001}}\right)\right].$$

$$S = \frac{1}{d}\left(\frac{1}{a_1} - \frac{1}{a_{4001}}\right) = \frac{1}{d}\left(\frac{a_{4001} - a_1}{a_1 a_{4001}}\right) = 10.$$

We know $a_{4001} - a_1 = (4001 - 1)d = 4000d$.

$$\frac{1}{d}\left(\frac{4000d}{a_1 a_{4001}}\right) = 10 \implies \frac{4000}{a_1 a_{4001}} = 10 \implies a_1 a_{4001} = 400.$$

Also given, $a_2 + a_{4000} = 50$.
For an AP, $a_2 + a_{4000} = a_1 + a_{4001}$. So, $a_1 + a_{4001} = 50$.

We need to find $|a_1 - a_{4001}|$.
We use the identity $(x - y)^2 = (x + y)^2 - 4xy$.
$(a_{4001} - a_1)^2 = (a_1 + a_{4001})^2 - 4a_1 a_{4001}$.
$(a_{4001} - a_1)^2 = (50)^2 - 4(400) = 2500 - 1600 = 900$.
$|a_{4001} - a_1| = \sqrt{900} = 30$.

The correct option is **(2)**.

57. Let a, ar, ar^2, be an infinite G.P. If $\sum_{n=0}^{\infty} ar^n = 57$ and $\sum_{n=0}^{\infty} a^3r^{3n} = 9747$, then $a+18r$ is equal to:

(1) 46 (2) 38 (3) 31 (4) 27

Solution:

Concept Used:

- Sum of an infinite GP: $S_\infty = \frac{a}{1-r}$ for $|r| < 1$.

$$\text{First sum: } \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} = 57 \quad \cdots (1).$$
$$\text{Second sum: } \sum_{n=0}^{\infty} a^3r^{3n} = \sum_{n=0}^{\infty} (ar^n)^3 = \frac{a^3}{1-r^3} = 9747 \quad \cdots (2).$$

From (1), $a = 57(1-r)$.

Substitute this into the cube of equation (1): $\left(\frac{a}{1-r}\right)^3 = 57^3$.

$$\frac{a^3}{(1-r)^3} = 57^3 \implies a^3 = 57^3(1-r)^3.$$

Substitute this expression for a^3 into equation (2):

$$\frac{57^3(1-r)^3}{1-r^3} = 9747.$$
$$\frac{57^3(1-r)^3}{(1-r)(1+r+r^2)} = 9747.$$
$$57^3\frac{(1-r)^2}{1+r+r^2} = 9747.$$
$$57^2\frac{(1-r)^2}{1+r+r^2} = \frac{9747}{57} = 171.$$
$$3249\frac{1-2r+r^2}{1+r+r^2} = 171.$$
$$\frac{3249}{171}\frac{1-2r+r^2}{1+r+r^2} = 1 \implies 19\frac{1-2r+r^2}{1+r+r^2} = 1.$$
$$19(1-2r+r^2) = 1+r+r^2.$$
$$19 - 38r + 19r^2 = 1 + r + r^2.$$
$$18r^2 - 39r + 18 = 0.$$
$$6r^2 - 13r + 6 = 0.$$
$$(2r-3)(3r-2) = 0.$$
$$r = 3/2 \text{ or } r = 2/3.$$

For an infinite GP sum to converge, $|r| < 1,$ so we must have $r = 2/3$.

Now find a using equation (1):

$a = 57(1-r) = 57(1-2/3) = 57(1/3) = 19.$

The required value is $a + 18r = 19 + 18(2/3) = 19 + 12 = 31.$

The correct option is **(3)**.

58. Given that $a_4 + a_8 + a_{12} + a_{16} = 224$, the sum of the first nineteen terms of the arithmetic progression $a_1, a_2, a_3, \ldots$ is equal to:

(1) 1540 (2) 1064 (3) 3125 (4) 1980

Solution:

Concept Used:

- Properties of an AP, specifically that the sum of terms equidistant from the beginning and end is constant.
- Sum of an AP: $S_n = \frac{n}{2}[2a + (n-1)d]$.

Hint:

- Express the given sum in terms of the first term 'a' and common difference 'd'.
- Simplify the expression to find a value for $a + 9d$, which is the 10th term.
- Use this value in the formula for S_{19}.

Let the AP have first term 'a' and common difference 'd'.
Given: $a_4 + a_8 + a_{12} + a_{16} = 224$.
$(a + 3d) + (a + 7d) + (a + 11d) + (a + 15d) = 224$.
$4a + (3 + 7 + 11 + 15)d = 224$.
$4a + 36d = 224$.
Divide by 4:
$a + 9d = 56$.
Note that $a + 9d$ is the 10th term, a_{10}.

We need to find the sum of the first 19 terms, S_{19}.

$$S_{19} = \frac{19}{2}[2a + (19-1)d] = \frac{19}{2}[2a + 18d].$$

$$S_{19} = \frac{19}{2} \cdot 2(a + 9d) = 19(a + 9d).$$

Substitute the value of $a + 9d$:
$S_{19} = 19 \times 56$.
$S_{19} = 19 \times (50 + 6) = 950 + 114 = 1064$.

The correct option is **(2)**.

59. The sum of the first 20 terms common to the series $3+7+11+15+...$ and $1+6+11+16+...$ is:

(1) 4000 (2) 4200 (3) 4220 (4) 4020

Solution:

Concept Used:

- Common terms of two APs form a new AP.
- The common difference of the new AP is LCM of the individual common differences.

Hint:

- Find the first common term by inspection.
- Find the common difference of the AP of common terms.
- Use the sum formula for an AP to find the sum of the first 20 common terms.

First series (S1): $3, 7, 11, 15, \ldots$ is an AP with $a_1 = 3, d_1 = 4$.
Second series (S2): $1, 6, 11, 16, \ldots$ is an AP with $a_2 = 1, d_2 = 5$.

By inspection, the first common term is 11.
The common difference of the AP formed by common terms is $d_c = \text{LCM}(d_1, d_2) = \text{LCM}(4, 5) = 20$.
So, the new AP of common terms has first term $a_c = 11$ and common difference $d_c = 20$.

We need to find the sum of the first 20 terms of this new AP.
$S_{20} = \frac{20}{2}[2a_c + (20-1)d_c]$.
$S_{20} = 10[2(11) + 19(20)]$.
$S_{20} = 10[22 + 380] = 10[402] = 4020$.

The correct option is **(4)**.

60. If the sum $\frac{3}{1^2} + \frac{5}{1^2+2^2} + \frac{7}{1^2+2^2+3^2} + ...+$ up to 20 terms is equal to $\frac{k}{21}$, then k is equal to:

(1) 240 (2) 120 (3) 60 (4) 180

Solution:

Concept Used:

- Sum of squares of first n natural numbers: $\sum n^2 = \frac{n(n+1)(2n+1)}{6}$.
- Method of difference / Telescoping series.

Hint:

- Find the general nth term (T_n) of the series.
- Simplify T_n and express it as a difference of two consecutive terms.
- Find the sum of 20 terms by observing the telescoping cancellation.

First, let's find the general term, T_n.
The numerator is an AP: $3, 5, 7, \ldots$ with nth term $3 + (n-1)2 = 2n + 1$.

The denominator is the sum of squares of first n natural numbers: $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

$$T_n = \frac{2n+1}{\frac{n(n+1)(2n+1)}{6}} = \frac{6}{n(n+1)}.$$

Using partial fractions:

$$T_n = 6\left(\frac{1}{n} - \frac{1}{n+1}\right).$$

The sum of the first 20 terms, $S_{20} = \sum_{n=1}^{20} T_n$.

$$S_{20} = 6\sum_{n=1}^{20}\left(\frac{1}{n} - \frac{1}{n+1}\right).$$

This is a telescoping series:

$$S_{20} = 6\left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{20} - \frac{1}{21}\right)\right].$$

$$S_{20} = 6\left(1 - \frac{1}{21}\right) = 6\left(\frac{20}{21}\right) = \frac{120}{21}.$$

We are given that the sum is $\frac{k}{21}$.

$$\frac{k}{21} = \frac{120}{21} \implies k = 120.$$

The correct option is **(2)**.

61. If $1, \log_3 \sqrt{(3^{1-x}+2)}, \log_3(4 \cdot 3^x - 1)$ are in arithmetic progression, then x equals:

(1) $\log_3 4$ (2) $1 - \log_3 4$ (3) $1 - \log_4 3$ (4) $\log_4 3$

Solution:

Concept Used:

- If a, b, c are in AP, then $2b = a + c$.
- Logarithm properties: $n \log a = \log a^n$, $\log a + \log b = \log(ab)$, $\log_b b = 1$.

Hint:

- Apply the condition for an AP.
- Use logarithm properties to simplify the equation.
- Let $y = 3^x$ to form a quadratic equation and solve for y.

Since the terms are in AP, we have:
$2 \log_3 \sqrt{3^{1-x}+2} = 1 + \log_3(4 \cdot 3^x - 1)$.
$\log_3 \left((\sqrt{3^{1-x}+2})^2\right) = \log_3 3 + \log_3(4 \cdot 3^x - 1)$.
$\log_3(3^{1-x}+2) = \log_3(3(4 \cdot 3^x - 1))$.
Equating the arguments of the logarithm:
$3^{1-x} + 2 = 3(4 \cdot 3^x - 1)$.
$\frac{3}{3^x} + 2 = 12 \cdot 3^x - 3$.
Let $y = 3^x$.
$\frac{3}{y} + 2 = 12y - 3 \implies \frac{3}{y} + 5 = 12y$.
Multiply by y: $3 + 5y = 12y^2$.
$12y^2 - 5y - 3 = 0$.
$12y^2 - 9y + 4y - 3 = 0$.
$3y(4y-3) + 1(4y-3) = 0 \implies (3y+1)(4y-3) = 0$.
$y = -1/3$ or $y = 3/4$.
Since $y = 3^x, y$ must be positive. So, $y = 3/4$.
$3^x = 3/4$.
$x = \log_3(3/4) = \log_3 3 - \log_3 4 = 1 - \log_3 4$.

The correct option is **(2)**.

62. The sum of the series $3 + 8 + 16 + 27 + 41 \ldots$ upto 20 terms is equal to:

(1) 4230 (2) 4430 (3) 4330 (4) 4500

Solution:

Concept Used:

- Method of differences for finding the nth term of a series.
- Summation formulas for standard series ($\sum n^2$, $\sum n$).

Hint:

- Let the sum be S_{20}. Find the differences between consecutive terms.
- If the differences are in AP, the nth term is a quadratic in n.
- Find the general term T_n and then find the sum $\sum_{n=1}^{20} T_n$.

Let the series be $S = 3 + 8 + 16 + 27 + 41 + \cdots + T_{20}$.
The differences between consecutive terms are:
$8 - 3 = 5, \quad 16 - 8 = 8, \quad 27 - 16 = 11, \quad 41 - 27 = 14, \ldots$
The first differences $5, 8, 11, 14, \ldots$ form an AP with a=5, d=3.
This means the nth term of the original series is a quadratic in n, i.e., $T_n = an^2 + bn + c$.
$T_1 = a + b + c = 3$.
$T_2 = 4a + 2b + c = 8$.
$T_3 = 9a + 3b + c = 16$.
$(T_2 - T_1) : 3a + b = 5$.
$(T_3 - T_2) : 5a + b = 8$.
Subtracting these two gives $2a = 3 \implies a = 3/2$.
$b = 5 - 3a = 5 - 3(3/2) = 5 - 9/2 = 1/2$.
$c = 3 - a - b = 3 - 3/2 - 1/2 = 3 - 2 = 1$.
$T_n = \frac{3}{2}n^2 + \frac{1}{2}n + 1$.

$$S_{20} = \sum_{n=1}^{20} T_n = \sum_{n=1}^{20} \left(\frac{3}{2}n^2 + \frac{1}{2}n + 1\right).$$
$$= \frac{3}{2}\sum n^2 + \frac{1}{2}\sum n + \sum 1.$$
$$= \frac{3}{2}\left(\frac{20(21)(41)}{6}\right) + \frac{1}{2}\left(\frac{20(21)}{2}\right) + 20.$$
$$= \frac{3}{2}(10 \cdot 7 \cdot 41) + \frac{1}{2}(10 \cdot 21) + 20.$$
$$= 3 \cdot 5 \cdot 7 \cdot 41 + 105 + 20 = 15 \cdot 287 + 125.$$
$$= 4305 + 125 = 4430.$$

The correct option is **(2)**.

63. The sum to the infinite terms of the series $\frac{5}{3^2 \cdot 7^2} + \frac{9}{7^2 \cdot 11^2} + \frac{13}{11^2 \cdot 15^2} + ...$ is:

(1) 1/8 (2) 1/36 (3) 1/54 (4) 1/72

Solution:

Concept Used:

- Method of difference / Telescoping series.
- Algebraic identity: $b^2 - a^2 = (b-a)(b+a)$.

Hint:

- Identify the general term T_n. The denominators involve squares of numbers in an AP.
- Express the numerator in terms of the factors in the denominator to facilitate cancellation.
- Decompose T_n as a difference of two terms.

The terms in the denominator are $3, 7, 11, 15, \ldots$ which is an AP with $a = 3, d = 4$.
The nth term is $3 + (n-1)4 = 4n - 1$.
The denominator of T_n is $(4n-1)^2(4(n+1)-1)^2 = (4n-1)^2(4n+3)^2$.
The numerator is $5, 9, 13, \ldots$ which is an AP with $a = 5, d = 4$.
The nth term is $5 + (n-1)4 = 4n + 1$.
$T_n = \dfrac{4n+1}{(4n-1)^2(4n+3)^2}$.
This is complex. Let's try to express the numerator as $(4n+3)^2 - (4n-1)^2$.
$(16n^2 + 24n + 9) - (16n^2 - 8n + 1) = 32n + 8 = 8(4n+1)$.
So, $4n + 1 = \frac{1}{8}((4n+3)^2 - (4n-1)^2)$.
$$T_n = \frac{1}{8}\frac{(4n+3)^2 - (4n-1)^2}{(4n-1)^2(4n+3)^2} = \frac{1}{8}\left(\frac{1}{(4n-1)^2} - \frac{1}{(4n+3)^2}\right).$$

The sum S is a telescoping series:
$$S = \frac{1}{8}\sum_{n=1}^{\infty}\left(\frac{1}{(4n-1)^2} - \frac{1}{(4n+3)^2}\right).$$
$n = 1 : \frac{1}{8}(\frac{1}{3^2} - \frac{1}{7^2})$.
$n = 2 : \frac{1}{8}(\frac{1}{7^2} - \frac{1}{11^2})$.
$n = 3 : \frac{1}{8}(\frac{1}{11^2} - \frac{1}{15^2})$.
...and so on.
As $n \to \infty$, the term $\dfrac{1}{(4n+3)^2} \to 0$.
$$S = \frac{1}{8}\left(\frac{1}{3^2} - 0\right) = \frac{1}{8} \times \frac{1}{9} = \frac{1}{72}.$$

The correct option is **(4)**.

64. The sum to infinity of the series $1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + ...$ is:

(1) 16/25 (2) 11/5 (3) 35/16 (4) 8/11

Solution:

Concept Used:

- Sum of an infinite Arithmetico-Geometric Progression (AGP).

Hint:

- The series is an AGP. Identify the AP part and the GP part.
- Use the formula $S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$ or derive it using the $S - rS$ method.

The given series is an AGP.
The AP part is $1, 4, 7, 10, \ldots$ with first term $a = 1$ and common difference $d = 3$.
The GP part is $1, 1/5, 1/5^2, \ldots$ with common ratio $r = 1/5$.

Let the sum be S.

$$S = 1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \ldots \quad \cdots (1)$$

$$\frac{1}{5}S = \quad \frac{1}{5} + \frac{4}{5^2} + \frac{7}{5^3} + \ldots \quad \cdots (2)$$

Subtract (2) from (1):

$$S(1 - \frac{1}{5}) = 1 + \left(\frac{4}{5} - \frac{1}{5}\right) + \left(\frac{7}{5^2} - \frac{4}{5^2}\right) + \ldots$$

$$\frac{4}{5}S = 1 + \frac{3}{5} + \frac{3}{5^2} + \frac{3}{5^3} + \ldots$$

$$\frac{4}{5}S = 1 + 3\left(\frac{1}{5} + \frac{1}{5^2} + \ldots\right).$$

The term in parenthesis is an infinite GP with first term 1/5 and ratio 1/5.

Sum of this GP is $\frac{1/5}{1 - 1/5} = \frac{1/5}{4/5} = \frac{1}{4}$.

$$\frac{4}{5}S = 1 + 3\left(\frac{1}{4}\right) = 1 + \frac{3}{4} = \frac{7}{4}.$$

$$S = \frac{7}{4} \times \frac{5}{4} = \frac{35}{16}.$$

The correct option is **(3)**.

65. The sum of the infinite series $\frac{1}{9} + \frac{1}{18} + \frac{1}{30} + \frac{1}{45} + \frac{1}{63} + ...$ is equal to:

(1) 1/3 (2) 1/4 (3) 1/5 (4) 2/3

Solution:

Concept Used:

- Recognizing patterns in series.
- Telescoping series.

Hint:

- Factorize the denominators to find a pattern.
- Note that the denominators are related to triangular numbers.

Let the series be S. Let's examine the denominators.
$9, 18, 30, 45, 63, \ldots$
Let's factor them: $9 = 3 \times 3, 18 = 3 \times 6, 30 = 3 \times 10, 45 = 3 \times 15, 63 = 3 \times 21, \ldots$
$S = \frac{1}{3}\left(\frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \frac{1}{21} + \ldots\right).$
The numbers $3, 6, 10, 15, 21, \ldots$ are triangular numbers, $T_n = \frac{n(n+1)}{2}$.
$T_2 = 3, T_3 = 6, T_4 = 10, \ldots$
So, the nth term of the inner series is $\frac{1}{T_{n+1}} = \frac{2}{(n+1)(n+2)}$.

$S = \frac{1}{3}\sum_{n=2}^{\infty} \frac{2}{n(n+1)}.$
Let's check the first term: n=2 gives $\frac{2}{2(3)} = 1/3$. Correct.
Let's check the second term: n=3 gives $\frac{2}{3(4)} = 1/6$. Correct.

$S = \frac{2}{3}\sum_{n=2}^{\infty} \frac{1}{n(n+1)} = \frac{2}{3}\sum_{n=2}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right).$
This is a telescoping series:
$S = \frac{2}{3}\left[(\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{4} - \frac{1}{5}) + \ldots\right].$
$S = \frac{2}{3}\left(\frac{1}{2}\right) = \frac{1}{3}.$

The correct option is **(1)**.

66. The sum of an infinite geometric series with positive terms is 3 and the sum of the cubes of its terms is $\frac{27}{19}$. Then the common ratio of this series is

(1) 4/9 (2) 2/9 (3) 2/3 (4) 1/3

Solution:

Concept Used:

- The sum of an infinite GP is $S_\infty = \frac{a}{1-r}$, for $|r| < 1$.
- If a GP is $a, ar, ar^2, \ldots$, the series of its cubes is $a^3, a^3r^3, a^3r^6, \ldots$, which is also a GP with first term a^3 and common ratio r^3.

Hint:

- Formulate two equations based on the two given sums.
- Solve the system of equations for the common ratio r.

Let the GP be $a, ar, ar^2, \ldots$

The sum of the series is $\dfrac{a}{1-r} = 3 \quad \cdots (1)$.

The series of the cubes is $a^3, a^3r^3, a^3r^6, \ldots$

This is a GP with first term a^3 and common ratio r^3.

The sum of this series is $\dfrac{a^3}{1-r^3} = \dfrac{27}{19} \quad \cdots (2)$.

From equation (1), cube both sides:

$$\left(\frac{a}{1-r}\right)^3 = 3^3 \implies \frac{a^3}{(1-r)^3} = 27 \implies a^3 = 27(1-r)^3.$$

Substitute this into equation (2):

$\dfrac{27(1-r)^3}{1-r^3} = \dfrac{27}{19}$.

$\dfrac{(1-r)^3}{(1-r)(1+r+r^2)} = \dfrac{1}{19}$.

$\dfrac{(1-r)^2}{1+r+r^2} = \dfrac{1}{19}$.

$19(1-2r+r^2) = 1+r+r^2$.

$19 - 38r + 19r^2 = 1 + r + r^2$.

$18r^2 - 39r + 18 = 0$.

Divide by 3: $6r^2 - 13r + 6 = 0$.

$6r^2 - 9r - 4r + 6 = 0 \implies 3r(2r-3) - 2(2r-3) = 0$.

$(3r-2)(2r-3) = 0$.

$r = 2/3$ or $r = 3/2$.

Since the sum is of an infinite series, we must have $|r| < 1$. Therefore, $r = 2/3$.

The correct option is **(3)**.

67. Let a_n be the n^{th} term of an A.P. If $S_n = a_1 + a_2 + a_3 + ... + a_n = 700$, $a_6 = 7$ and $S_7 = 7$ then a_n is equal to

(1) 56 (2) 65 (3) 64 (4) 70

Solution:

Concept Used:

- nth term of an AP: $a_k = a_1 + (k-1)d$.
- Sum of n terms of an AP: $S_k = \frac{k}{2}[2a_1 + (k-1)d]$.

Hint:

- Use the given values of a_6 and S_7 to form a system of two linear equations in a_1 and d.
- Solve for a_1 and d.
- Use the value of $S_n = 700$ to find the value of n.
- Finally, calculate the term a_n.

$$\text{Given } a_6 = 7 \implies a_1 + 5d = 7 \quad \cdots(1).$$
$$\text{Given } S_7 = 7 \implies \frac{7}{2}[2a_1 + (7-1)d] = 7.$$
$$\frac{1}{2}[2a_1 + 6d] = 1 \implies a_1 + 3d = 1 \quad \cdots(2).$$

Subtract equation (2) from (1):
$(a_1 + 5d) - (a_1 + 3d) = 7 - 1 \implies 2d = 6 \implies d = 3.$
Substitute d=3 into (2): $a_1 + 3(3) = 1 \implies a_1 = 1 - 9 = -8.$

Now use $S_n = 700$ to find n.
$$\frac{n}{2}[2a_1 + (n-1)d] = 700.$$
$$\frac{n}{2}[2(-8) + (n-1)3] = 700.$$
$$n[-16 + 3n - 3] = 1400.$$
$$n(3n - 19) = 1400 \implies 3n^2 - 19n - 1400 = 0.$$
Solving the quadratic for n:
$$n = \frac{-(-19) \pm \sqrt{(-19)^2 - 4(3)(-1400)}}{2(3)} = \frac{19 \pm \sqrt{361 + 16800}}{6} = \frac{19 \pm \sqrt{17161}}{6}.$$
$$\sqrt{17161} = 131.$$
$$n = \frac{19 \pm 131}{6}. \text{ Since n must be positive, } n = \frac{19 + 131}{6} = \frac{150}{6} = 25.$$

The question asks for a_n, which is a_{25}.
$$a_{25} = a_1 + 24d = -8 + 24(3) = -8 + 72 = 64.$$

The correct option is **(3)**.

68. If the sum of the second, fourth and sixth terms of a GP of positive terms is 21 and the sum of its eighth, tenth and twelth terms is 15309, then the sum of its first nine terms is:

(1) 760 (2) 755 (3) 750 (4) 757

Solution:

Concept Used:

- nth term of a GP: $a_n = ar^{n-1}$.
- Sum of n terms of a GP: $S_n = \frac{a(r^n-1)}{r-1}$.

Hint:

- Write the two given conditions as equations in terms of 'a' and 'r'.
- Divide the second equation by the first to eliminate 'a' and solve for 'r'.
- Substitute 'r' back into the first equation to find 'a'.
- Calculate the sum of the first nine terms.

Let the first term be 'a' and the common ratio be 'r'. Since terms are positive, $a > 0, r > 0$.
Given: $a_2 + a_4 + a_6 = 21$.
$ar + ar^3 + ar^5 = 21 \implies ar(1 + r^2 + r^4) = 21 \quad \cdots (1)$.

Given: $a_8 + a_{10} + a_{12} = 15309$.
$ar^7 + ar^9 + ar^{11} = 15309 \implies ar^7(1 + r^2 + r^4) = 15309 \quad \cdots (2)$.

Divide equation (2) by equation (1):
$$\frac{ar^7(1 + r^2 + r^4)}{ar(1 + r^2 + r^4)} = \frac{15309}{21}.$$
$r^6 = 729$.
Since r must be positive, $r = (729)^{1/6} = (3^6)^{1/6} = 3$.

Substitute r=3 into equation (1):
$a(3)(1 + 3^2 + 3^4) = 21$.
$3a(1 + 9 + 81) = 21 \implies 3a(91) = 21$.
$$a = \frac{21}{3 \times 91} = \frac{7}{91} = \frac{1}{13}.$$

Now find the sum of the first nine terms, S_9 :
$$S_9 = \frac{a(r^9 - 1)}{r - 1} = \frac{\frac{1}{13}(3^9 - 1)}{3 - 1}.$$
$$S_9 = \frac{1}{13} \cdot \frac{19683 - 1}{2} = \frac{19682}{26}.$$
$S_9 = 757$.

The correct option is **(4)**.

69. Let $a_1, a_2, a_3, ...$ be a G.P. of increasing positive numbers. If $a_3 a_5 = 729$ and $a_2 + a_4 = \frac{111}{4}$ then $24(a_1 + a_2 + a_3)$ is equal to

(1) 131 (2) 130 (3) 129 (4) 128

Solution:

Concept Used:

- Properties of a GP: $a_n = ar^{n-1}$, and $a_i a_j = a_k a_l$ if $i + j = k + l$.

Hint:

- Use the property $a_3 a_5 = a_4^2$ to find the 4th term.
- Use the second given condition to find the 2nd term.
- Find the common ratio 'r' and the first term 'a'.
- Calculate the required expression.

Let the first term be 'a' and the common ratio be 'r'. Since it's an increasing GP of positive numbers, a
Given $a_3 a_5 = 729$.
In a GP, $a_3 a_5 = a_4^2$. So, $a_4^2 = 729 \implies a_4 = 27$.
$a_4 = ar^3 = 27$.

Given $a_2 + a_4 = 111/4$.
$a_2 + 27 = 111/4 \implies a_2 = \frac{111}{4} - 27 = \frac{111 - 108}{4} = \frac{3}{4}$.
$a_2 = ar = 3/4$.

Now, find r:
$\frac{a_4}{a_2} = \frac{ar^3}{ar} = r^2$.
$r^2 = \frac{27}{3/4} = 27 \times \frac{4}{3} = 9 \times 4 = 36$.
$r = 6$ (since the GP is increasing).

Find a:
$a = a_2/r = (3/4)/6 = 3/24 = 1/8$.

We need to find $24(a_1 + a_2 + a_3)$.
$a_1 = 1/8$.
$a_2 = 3/4 = 6/8$.
$a_3 = a_2 \times r = (3/4) \times 6 = 18/4 = 9/2 = 36/8$.
$a_1 + a_2 + a_3 = \frac{1 + 6 + 36}{8} = \frac{43}{8}$.
$24(a_1 + a_2 + a_3) = 24 \times \frac{43}{8} = 3 \times 43 = 129$.

The correct option is **(3)**.

70. If $7 = 5 + \frac{1}{7}(5+\alpha) + \frac{1}{7^2}(5+2\alpha) + \frac{1}{7^3}(5+3\alpha) + ...$, then the value of α is

(1) 6/7 (2) 6 (3) 1/7 (4) 1

Solution:

Concept Used:

- Sum of an infinite GP and an infinite AGP.

$$7 - 5 = \frac{5+\alpha}{7} + \frac{5+2\alpha}{7^2} + \frac{5+3\alpha}{7^3} + \dots$$
$$2 = \sum_{n=1}^{\infty} \frac{5+n\alpha}{7^n}.$$

Separate the sum into two parts:

$$2 = \sum_{n=1}^{\infty} \frac{5}{7^n} + \sum_{n=1}^{\infty} \frac{n\alpha}{7^n}.$$
$$2 = 5\left(\sum_{n=1}^{\infty} \frac{1}{7^n}\right) + \alpha\left(\sum_{n=1}^{\infty} \frac{n}{7^n}\right).$$

Part 1: Sum of a GP.

$$\sum_{n=1}^{\infty} \frac{1}{7^n} = \frac{1}{7} + \frac{1}{7^2} + \cdots = \frac{a'}{1-r'} = \frac{1/7}{1-1/7} = \frac{1/7}{6/7} = \frac{1}{6}.$$

Part 2: Sum of an AGP.

$$\text{Let } P = \sum_{n=1}^{\infty} \frac{n}{7^n} = \frac{1}{7} + \frac{2}{7^2} + \frac{3}{7^3} + \dots$$
$$\frac{1}{7}P = \quad \frac{1}{7^2} + \frac{2}{7^3} + \dots$$
$$P - \frac{1}{7}P = \frac{6}{7}P = \frac{1}{7} + \frac{1}{7^2} + \frac{1}{7^3} + \cdots = \frac{1}{6}.$$
$$\frac{6}{7}P = \frac{1}{6} \implies P = \frac{7}{36}.$$

Substitute these sums back into the main equation:

$$2 = 5\left(\frac{1}{6}\right) + \alpha\left(\frac{7}{36}\right).$$
$$2 = \frac{5}{6} + \frac{7\alpha}{36}.$$
$$2 - \frac{5}{6} = \frac{7\alpha}{36}.$$
$$\frac{12-5}{6} = \frac{7}{6} = \frac{7\alpha}{36}.$$
$$\frac{1}{6} = \frac{\alpha}{36} \implies \alpha = 6.$$

The correct option is **(2)**.

SECTION-B

71. A sequence of equilateral triangles is drawn. The altitude of each is $\sqrt{3}$ times the altitude of the preceding triangle, the difference between the area of the first triangle and the sixth triangle is 968 $\sqrt{3}$ square unit. The perimeter of the first triangle is

Solution:

Concept Used:

- Properties of equilateral triangles: Area $A = \frac{\sqrt{3}}{4}s^2$, Altitude $h = \frac{\sqrt{3}}{2}s$.
- Relation between Area and Altitude: $s = \frac{2h}{\sqrt{3}} \implies A = \frac{\sqrt{3}}{4}\left(\frac{4h^2}{3}\right) = \frac{h^2}{\sqrt{3}}$.
- Geometric Progression.

Let h_n be the altitude of the nth triangle.
Given $h_{n+1} = \sqrt{3}h_n$. This means the altitudes form a GP with common ratio $r_h = \sqrt{3}$.
$h_6 = h_1(\sqrt{3})^{6-1} = h_1(\sqrt{3})^5 = 9\sqrt{3}h_1$.

Let A_n be the area of the nth triangle. $A_n = \dfrac{h_n^2}{\sqrt{3}}$.
The ratio of areas is $\dfrac{A_{n+1}}{A_n} = \dfrac{h_{n+1}^2/\sqrt{3}}{h_n^2/\sqrt{3}} = \left(\dfrac{h_{n+1}}{h_n}\right)^2 = (\sqrt{3})^2 = 3$.
The areas form a GP with common ratio $r_A = 3$.

Given that the difference is $968\sqrt{3}$. Since the altitude increases, the area increases.
$A_6 - A_1 = 968\sqrt{3}$.
$A_1(r_A)^{6-1} - A_1 = 968\sqrt{3}$.
$A_1(3^5 - 1) = 968\sqrt{3}$.
$A_1(243 - 1) = 968\sqrt{3}$.
$242A_1 = 968\sqrt{3} \implies A_1 = \dfrac{968\sqrt{3}}{242} = 4\sqrt{3}$.

Now find the side of the first triangle, s_1.
$A_1 = \dfrac{\sqrt{3}}{4}s_1^2 = 4\sqrt{3}$.
$s_1^2 = 16 \implies s_1 = 4$.

The perimeter of the first triangle is $3s_1 = 3 \times 4 = 12$.

The answer is **12**.

72. If rth term of a series is $(2r+1)2^{-r}$, then sum of its infinite terms is:

Solution:

Concept Used:

- Sum of an infinite Arithmetico-Geometric Progression (AGP).

Hint:

- Write out the first few terms of the series to identify it as an AGP.
- Use the $S - rS$ method to find the sum.

The rth term is $T_r = (2r+1)2^{-r} = \frac{2r+1}{2^r}$.
Let the sum be S, assuming the series starts from r=1.
$S = \frac{3}{2} + \frac{5}{4} + \frac{7}{8} + \frac{9}{16} + \dots \quad \cdots (1)$
This is an AGP with common ratio $r = 1/2$.
$\frac{1}{2}S = \quad \frac{3}{4} + \frac{5}{8} + \frac{7}{16} + \dots \quad \cdots (2)$

Subtract (2) from (1):
$S - \frac{1}{2}S = \frac{1}{2}S = \frac{3}{2} + \left(\frac{5}{4} - \frac{3}{4}\right) + \left(\frac{7}{8} - \frac{5}{8}\right) + \dots$
$\frac{1}{2}S = \frac{3}{2} + \frac{2}{4} + \frac{2}{8} + \frac{2}{16} + \dots$
$\frac{1}{2}S = \frac{3}{2} + 2\left(\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots\right)$.
The term in the parenthesis is an infinite GP with first term $a' = 1/4$ and ratio $r' = 1/2$.
Sum of this GP is $S'_{GP} = \frac{a'}{1-r'} = \frac{1/4}{1-1/2} = \frac{1/4}{1/2} = \frac{1}{2}$.

$\frac{1}{2}S = \frac{3}{2} + 2\left(\frac{1}{2}\right) = \frac{3}{2} + 1 = \frac{5}{2}$.
$S = 5$.

The answer is **5**.

73. If $\frac{5+9+13+...........n \text{ terms}}{7+9+11+......(n+1) \text{ terms}} = \frac{17}{16}$, then n is equal to:

Solution:

Concept Used:

- Sum of an Arithmetic Progression (AP): $S_k = \frac{k}{2}[2a + (k-1)d]$.

Hint:

- Identify the first term and common difference for both the numerator and denominator series.
- Apply the sum formula to both series.
- Solve the resulting equation for n.

Numerator: An AP with first term $a_1 = 5$, common difference $d_1 = 4$, and number of terms $k_1 = n$.
$S_{num} = \frac{n}{2}[2(5) + (n-1)4] = \frac{n}{2}[10 + 4n - 4] = \frac{n}{2}[4n + 6] = n(2n + 3)$.

Denominator: An AP with first term $a_2 = 7$, common difference $d_2 = 2$, and number of terms $k_2 = n +$
$S_{den} = \frac{n+1}{2}[2(7) + (n+1-1)2] = \frac{n+1}{2}[14 + 2n] = (n+1)(n+7)$.

The given equation is:
$\frac{n(2n+3)}{(n+1)(n+7)} = \frac{17}{16}$.
$16n(2n+3) = 17(n+1)(n+7)$.
$32n^2 + 48n = 17(n^2 + 8n + 7)$.
$32n^2 + 48n = 17n^2 + 136n + 119$.
$15n^2 - 88n - 119 = 0$.

Solving the quadratic equation for n:
$n = \frac{-(-88) \pm \sqrt{(-88)^2 - 4(15)(-119)}}{2(15)} = \frac{88 \pm \sqrt{7744 + 7140}}{30} = \frac{88 \pm \sqrt{14884}}{30}$.
We find that $\sqrt{14884} = 122$.
$n = \frac{88 \pm 122}{30}$.
Since n must be a positive integer (number of terms), we take the positive root.
$n = \frac{88 + 122}{30} = \frac{210}{30} = 7$.

The answer is **7**.

74. If $S = \sum_{r=1}^{90} \frac{r}{(r^4+r^2+1)}$; then the value of $\frac{8191S}{9}$ must be

Solution:

Concept Used:

- Telescoping series method.
- Factorization of $x^4 + x^2 + 1 = (x^2+1)^2 - x^2 = (x^2 - x + 1)(x^2 + x + 1)$.

Let the general term be $T_r = \dfrac{r}{r^4 + r^2 + 1}$.
The denominator can be factored as $(r^2 - r + 1)(r^2 + r + 1)$.
$$T_r = \frac{r}{(r^2 - r + 1)(r^2 + r + 1)}.$$

We can express T_r using partial fractions. Note that $(r^2 + r + 1) - (r^2 - r + 1) = 2r$.
$$T_r = \frac{1}{2} \cdot \frac{2r}{(r^2 - r + 1)(r^2 + r + 1)} = \frac{1}{2}\left[\frac{(r^2 + r + 1) - (r^2 - r + 1)}{(r^2 - r + 1)(r^2 + r + 1)}\right].$$
$$T_r = \frac{1}{2}\left[\frac{1}{r^2 - r + 1} - \frac{1}{r^2 + r + 1}\right].$$

This is a telescoping series. Let $f(r) = \dfrac{1}{r^2 - r + 1}$.
Then $f(r+1) = \dfrac{1}{(r+1)^2 - (r+1) + 1} = \dfrac{1}{r^2 + 2r + 1 - r - 1 + 1} = \dfrac{1}{r^2 + r + 1}$.
So, $T_r = \dfrac{1}{2}[f(r) - f(r+1)]$.

$$S = \sum_{r=1}^{90} T_r = \frac{1}{2}\sum_{r=1}^{90}[f(r) - f(r+1)].$$
$$S = \frac{1}{2}[(f(1) - f(2)) + (f(2) - f(3)) + \cdots + (f(90) - f(91))].$$
$$S = \frac{1}{2}[f(1) - f(91)].$$
$$f(1) = \frac{1}{1^2 - 1 + 1} = 1.$$
$$f(91) = \frac{1}{91^2 - 91 + 1} = \frac{1}{8281 - 90} = \frac{1}{8191}.$$
$$S = \frac{1}{2}\left(1 - \frac{1}{8191}\right) = \frac{1}{2}\left(\frac{8190}{8191}\right) = \frac{4095}{8191}.$$

The value to find is $\dfrac{8191S}{9} = \dfrac{8191}{9} \times \dfrac{4095}{8191} = \dfrac{4095}{9}$.
$= 455$.

The answer is **455**.

75. If $\{a_i\}_{i=1}^{n}$ where n is an even integer, is an arithmetic progression with common difference 1, and $\sum_{i=1}^{n} a_i = 192, \sum_{i=1}^{n/2} a_{2i} = 120$, then n is equal to

Solution:

Concept Used:

- Sum of an AP.
- A subsequence with evenly spaced terms from an AP is also an AP.

Hint:

- Set up two equations based on the two given sums.
- The second sum is the sum of an AP consisting of the even terms $a_2, a_4, \ldots, a_n$.
- Solve the system of two equations for the two unknowns, a_1 and n.

Let the first term be $a_1 = a,$ and the common difference is $d = 1$.

First condition: $\sum_{i=1}^{n} a_i = 192.$

$S_n = \frac{n}{2}[2a_1 + (n-1)d] = \frac{n}{2}[2a + (n-1)(1)] = 192.$

$n(2a + n - 1) = 384 \quad \cdots (1).$

Second condition: $\sum_{i=1}^{n/2} a_{2i} = 120.$

This is the sum of the series $a_2, a_4, a_6, \ldots, a_n$.

This is an AP with $n/2$ terms.

First term of this new AP is $a'_1 = a_2 = a_1 + d = a + 1$.

Common difference of this new AP is $d' = a_4 - a_2 = (a + 3d) - (a + d) = 2d = 2$.

Sum of this AP is $S'_{n/2} = \frac{n/2}{2}[2a'_1 + (\frac{n}{2} - 1)d']$.

$120 = \frac{n}{4}[2(a + 1) + (\frac{n}{2} - 1)2]$.

$120 = \frac{n}{4}[2a + 2 + n - 2] = \frac{n}{4}(2a + n)$.

$n(2a + n) = 480 \quad \cdots (2).$

We have a system of two equations:

$(1) : 2an + n^2 - n = 384$

$(2) : 2an + n^2 = 480$

Substitute $(2an + n^2)$ from (2) into (1):

$480 - n = 384.$

$n = 480 - 384 = 96.$

The answer is **96**.