Hi everyone,

The real learning in mathematics happens when you actively engage with a problem, explore different methods, and work through challenges. Therefore, we strongly encourage you to use this solution key responsibly.

Please attempt all the problems on your own first, giving them your best and most honest effort. These solutions are to help you get unstuck on a problem after you have already tried your best.

Your effort and dedication are the true keys to success.

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Syllabus: Functions, Sets & Relations

Sub: Mathematics

CT-14 JEE Main Solution

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1. The real valued function $f(x) = \frac{cosec^{-1}x}{\sqrt{x-[x]}}$, where [x] denotes the greatest integer less than or equal to x, is defined for all x belonging to:

(A) all reals except integers
(B) all non-integers except the interval [-1,1]
(C) all integers except 0, -1, 1
(D) all reals except the Interval [-1,1]

Answer: (B)
Solution:

Let $f(x) = \dfrac{g(x)}{h(x)}$ where $g(x) = \csc^{-1}(x)$ and $h(x) = \sqrt{x-[x]}$.

1. Domain of the numerator $g(x) = \csc^{-1}(x)$:
The domain of $\csc^{-1}(x)$ is $|x| \geq 1$, which is $x \in (-\infty, -1] \cup [1, \infty)$.

2. Domain of the denominator $h(x) = \sqrt{x-[x]}$:
We know that $x - [x]$ is the fractional part of x, denoted as $\{x\}$.
So, $h(x) = \sqrt{\{x\}}$.
For the square root to be defined, $\{x\} \geq 0$, which is always true.
However, since it is in the denominator, $h(x) \neq 0$, so $\{x\} \neq 0$.
$\{x\} = 0$ if and only if x is an integer.
Therefore, the domain for the denominator is x is not an integer ($x \notin \mathbb{Z}$).

3. Overall Domain (Intersection):
We must satisfy both conditions. We take the domain from (1) and remove all integers from it.
Domain $= ((-\infty, -1] \cup [1, \infty)) - \mathbb{Z}$
This is $(-\infty, -1) \cup (1, \infty)$.
This can be described as "all non-integers except the interval [-1,1]".

2. If the domain of the function $f(x) = sin^{-1}(\frac{3x-22}{2x-19}) + log_e(\frac{3x^2-8x+5}{x^2-3x-10})$ is $(\alpha, \beta]$, then $3\alpha + 10\beta$ is equal to:

(A) 100
(B) 95
(C) 97
(D) 98

Answer: (C)
Solution:

Let $f(x) = g(x) + h(x)$. The domain is the intersection of D_g and D_h.

$$1.\ \text{Domain of } g(x) = \sin^{-1}(\frac{3x-22}{2x-19}):$$

$$\text{We need } -1 \le \frac{3x-22}{2x-19} \le 1. \text{ We solve this in two parts.}$$

$$\text{Part (a): } \frac{3x-22}{2x-19} \le 1 \implies \frac{3x-22-(2x-19)}{2x-19} \le 0 \implies \frac{x-3}{2x-19} \le 0.$$

$$\text{Critical points are } x = 3, x = 19/2. \text{ Solution is } [3, 19/2).$$

$$\text{Part (b): } \frac{3x-22}{2x-19} \ge -1 \implies \frac{3x-22+(2x-19)}{2x-19} \ge 0 \implies \frac{5x-41}{2x-19} \ge 0.$$

$$\text{Critical points are } x = 41/5, x = 19/2. \text{ Solution is } (-\infty, 41/5] \cup (19/2, \infty).$$

$$\text{Intersection (a) and (b): } D_g = [3, 41/5]. \quad \text{(Note: } 41/5 = 8.2 \text{ and } 19/2 = 9.5)$$

$$2.\ \text{Domain of } h(x) = \log_e(\frac{3x^2-8x+5}{x^2-3x-10}):$$

$$\text{We need } \frac{3x^2-8x+5}{x^2-3x-10} > 0 \implies \frac{(3x-5)(x-1)}{(x-5)(x+2)} > 0.$$

$$\text{Critical points are } -2, 1, 5/3, 5.$$

$$\text{Solution (using sign chart): } D_h = (-\infty, -2) \cup (1, 5/3) \cup (5, \infty).$$

$$3.\ \text{Overall Domain } D_f = D_g \cap D_h:$$

$$D_g = [3, 8.2]$$

$$D_h = (-\infty, -2) \cup (1, 1.66...) \cup (5, \infty)$$

$$\text{The intersection is } (5, 8.2], \text{ which is } (5, 41/5].$$

$$\text{Thus, } \alpha = 5 \text{ and } \beta = 41/5.$$

$$3\alpha + 10\beta = 3(5) + 10(41/5) = 15 + 2(41) = 15 + 82 = 97.$$

3. The minimum value of $2^{sin\ x} + 2^{cos\ x}$ is:

(A) $2^{-1+\frac{1}{\sqrt{2}}}$ (B) $2^{-1+\sqrt{2}}$ (C) $2^{1-\sqrt{2}}$ (D) $2^{1-\frac{1}{\sqrt{2}}}$

Answer: (D)
Solution:

$$\text{Let } f(x) = 2^{\sin x} + 2^{\cos x}.$$

$$\text{By A.M. } \ge \text{ G.M.:}$$

$$\frac{2^{\sin x} + 2^{\cos x}}{2} \ge \sqrt{2^{\sin x} \cdot 2^{\cos x}} = \sqrt{2^{\sin x + \cos x}} = 2^{\frac{\sin x + \cos x}{2}}.$$

$$\text{This gives } f(x) \ge 2^{1+\frac{\sin x + \cos x}{2}}.$$

$$\text{The minimum of } f(x) \text{ will occur where } \sin x + \cos x \text{ is minimum.}$$

$$\text{We know the range of } \sin x + \cos x \text{ is } [-\sqrt{2}, \sqrt{2}].$$

$$\text{The minimum value of } \sin x + \cos x \text{ is } -\sqrt{2}.$$

$$\text{This minimum is achieved when } \sin x = \cos x = -1/\sqrt{2} \text{ (e.g., } x = 5\pi/4).$$

$$\text{At this point, the A.M. = G.M., so we can find the minimum value by substitution:}$$

$$f_{min} = 2^{-1/\sqrt{2}} + 2^{-1/\sqrt{2}}$$

$$= 2 \cdot (2^{-1/\sqrt{2}})$$

$$= 2^1 \cdot 2^{-1/\sqrt{2}} = 2^{1-1/\sqrt{2}}.$$

4. Let $f : R \to R$ be defined by $f(x) = \frac{x}{1+x^2}$, $x \in R$. Then the range of f is:

(A) $[-\frac{1}{2}, \frac{1}{2}]$
(B) R – [-1,1]
(C) $R - [-\frac{1}{2}, \frac{1}{2}]$
(D) (1, 1) – 0

Answer: (A)
Solution:

$$\text{Let } y = f(x) = \frac{x}{1+x^2}.$$
We solve for x in terms of y.
$$y(1+x^2) = x$$
$$y + yx^2 = x$$
$$yx^2 - x + y = 0.$$
This is a quadratic equation in x.
Case 1: $y = 0$. The equation becomes $-x = 0 \implies x = 0$. So $y = 0$ is in the range.
Case 2: $y \neq 0$. For x to be real, the discriminant D must be ≥ 0.
$$D = b^2 - 4ac = (-1)^2 - 4(y)(y) \geq 0$$
$$1 - 4y^2 \geq 0$$
$$4y^2 \leq 1$$
$$y^2 \leq \frac{1}{4}$$
This implies $-\frac{1}{2} \leq y \leq \frac{1}{2}$.
Combining Case 1 and Case 2, the range is $[-\frac{1}{2}, \frac{1}{2}]$.

5. Let $A = \{x \in R : x \text{ is not a positive integer}\}$. Define a function $f : A \to R$ as $f(x) = \frac{2x}{x-1}$ then f is:

(A) Injective but not surjective
(B) Not injective
(C) Surjective but not injective
(D) Neither injective nor surjective

Answer: (A)
Solution:

Domain $A = \mathbb{R} - \{1, 2, 3, \ldots\}$. Codomain $= \mathbb{R}$.

1. Check for Injectivity (One-to-one):
Let $f(x_1) = f(x_2)$ for $x_1, x_2 \in A$.
$$\frac{2x_1}{x_1 - 1} = \frac{2x_2}{x_2 - 1}$$
$$x_1(x_2 - 1) = x_2(x_1 - 1)$$
$$x_1x_2 - x_1 = x_1x_2 - x_2$$
$$-x_1 = -x_2 \implies x_1 = x_2.$$
The function is injective.

2. Check for Surjectivity (Onto):
We find the range of $f(x)$. Let $y = \frac{2x}{x-1}$.
$$y(x-1) = 2x$$
$$yx - y = 2x$$
$$yx - 2x = y$$
$$x(y-2) = y \implies x = \frac{y}{y-2}.$$

Since x can be any real number except 2, the range is $\mathbb{R} - \{2\}$.
The codomain is given as $\mathbb{R}$.
Since Range $(\mathbb{R} - \{2\}) \neq$ Codomain $(\mathbb{R})$, the function is not surjective.

Therefore, the function is injective but not surjective.

6. The total number of functions, $f : \{1, 2, 3, 4\} \to \{1, 2, 3, 4, 5, 6\}$ such that $f(1) + f(2) = f(3)$, is equal to
(A) 60 (B) 90 (C) 108 (D) 126

Answer: (B)
Solution:

The domain is $A = \{1, 2, 3, 4\}$ and codomain is $B = \{1, 2, 3, 4, 5, 6\}$.
The condition is $f(1) + f(2) = f(3)$.
We need to find pairs $(f(1), f(2))$ from $B \times B$ whose sum $f(3)$ is also in B.
Let's list the possible cases:
If $f(1) = 1,$ then $f(3) = 1 + f(2)$. Since $f(3) \leq 6, f(2) \leq 5$. So $f(2) \in \{1, 2, 3, 4, 5\}$. (5 ways)
If $f(1) = 2,$ then $f(3) = 2 + f(2)$. Since $f(3) \leq 6, f(2) \leq 4$. So $f(2) \in \{1, 2, 3, 4\}$. (4 ways)
If $f(1) = 3,$ then $f(3) = 3 + f(2)$. Since $f(3) \leq 6, f(2) \leq 3$. So $f(2) \in \{1, 2, 3\}$. (3 ways)
If $f(1) = 4,$ then $f(3) = 4 + f(2)$. Since $f(3) \leq 6, f(2) \leq 2$. So $f(2) \in \{1, 2\}$. (2 ways)
If $f(1) = 5,$ then $f(3) = 5 + f(2)$. Since $f(3) \leq 6, f(2) \leq 1$. So $f(2) \in \{1\}$. (1 way)
If $f(1) = 6,$ then $f(3) = 6 + f(2)$. This is impossible as $f(2) \geq 1,$ so $f(3) \geq 7$. (0 ways)

Total number of ways to choose $f(1), f(2),$ and $f(3)$ is $5 + 4 + 3 + 2 + 1 = 15$.
The value of $f(4)$ is independent and can be any of the 6 values in B.
Total number of functions $=$ (Ways for $f(1), f(2), f(3)$) $\times$ (Ways for $f(4)$)
$= 15 \times 6 = 90$.

7. Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 4, 9, 16\}$. Then the number of many-one functions $f : A \to B$ such that $1 \in f(A)$ is equal to:
(A) 151 (B) 139 (C) 163 (D) 127

Answer: (A)
Solution:

We have $n(A) = 4$ and $n(B) = 4$.
Total number of functions $f : A \to B$ is $n(B)^{n(A)} = 4^4 = 256$.
Number of one-one functions $f : A \to B$ is $4! = 24$.
Number of many-one functions $=$ (Total functions) $-$ (One-one functions)
$= 256 - 24 = 232$.

Now, we find the number of many-one functions that do NOT satisfy the condition,
i.e., functions where $1 \notin f(A)$ (1 is not in the range).
Let this set of functions be E. For $f \in E,$ the range is a subset of $B' = \{4, 9, 16\}$.
Total functions $f : A \to B'$ is $n(B')^{n(A)} = 3^4 = 81$.
Number of one-one functions $f : A \to B'$ is 0 (by pigeonhole principle, as $n(A) > n(B')$).

So, the number of many-one functions in $E = 81 - 0 = 81$.

The required number of functions is:
(Total many-one functions) – (Many-one functions where $1 \notin f(A)$)
$= 232 - 81 = 151$.

8. If a function f satisfies $f(m+n) = f(m) + f(n)$ for all m, $n \in N$ and $f(1) = 1$, then the largest natural number λ such that $\Sigma_{k=1}^{2022} f(\lambda + k) \le (2022)^2$ is equal to

Answer: 1010
Solution:

Given $f(m+n) = f(m) + f(n)$ and $f(1) = 1$.
This is Cauchy's functional equation over $\mathbb{N}$.
$f(2) = f(1+1) = f(1) + f(1) = 2$.
$f(3) = f(2+1) = f(2) + f(1) = 2 + 1 = 3$.
By induction, we get $f(x) = x$ for all $x \in \mathbb{N}$.

Now we evaluate the sum:

$$\sum_{k=1}^{2022} f(\lambda + k) = \sum_{k=1}^{2022} (\lambda + k)$$
$$= \sum_{k=1}^{2022} \lambda + \sum_{k=1}^{2022} k$$
$$= (\lambda + \lambda + \cdots + \lambda) + (1 + 2 + \cdots + 2022)$$
$$= 2022\lambda + \frac{2022(2023)}{2}$$
$$= 2022\lambda + 1011 \cdot 2023.$$

We are given the inequality:

$$2022\lambda + 1011 \cdot 2023 \le (2022)^2$$
$$2022\lambda \le (2022)^2 - 1011 \cdot 2023$$
$$2022\lambda \le (2 \cdot 1011) \cdot 2022 - 1011 \cdot 2023$$
$$2022\lambda \le 1011 \cdot (2 \cdot 2022 - 2023)$$
$$2022\lambda \le 1011 \cdot (4044 - 2023)$$
$$2022\lambda \le 1011 \cdot (2021)$$
$$\lambda \le \frac{1011 \cdot 2021}{2022} = \frac{1011 \cdot 2021}{2 \cdot 1011} = \frac{2021}{2}$$
$$\lambda \le 1010.5.$$

The largest natural number (integer) λ is 1010.

9. For the differentiable function $f : \mathbb{R} - \{0\} \to \mathbb{R}$ let $3f(x) + 2f(\frac{1}{x}) = \frac{1}{x} - 10$, then $|f(3) + f'(\frac{1}{4})|$ is

(A) $\frac{33}{5}$ (B) 8 (C) $\frac{29}{5}$ (D) 13

Answer: (D)
Solution:

We are given the functional equation:

$$(1) \quad 3f(x) + 2f(1/x) = 1/x - 10$$

Replace x with $1/x$ to get a second equation:

$$(2) \quad 3f(1/x) + 2f(x) = x - 10$$

We solve this system for $f(x)$. Multiply (1) by 3 and (2) by 2:

$$(1') \quad 9f(x) + 6f(1/x) = 3/x - 30$$

$$(2') \quad 4f(x) + 6f(1/x) = 2x - 20$$

Subtract (2') from (1'):

$$(9f(x) - 4f(x)) + 0 = (3/x - 30) - (2x - 20)$$

$$5f(x) = 3/x - 2x - 10$$

$$f(x) = \frac{3}{5x} - \frac{2x}{5} - 2.$$

Now, find the required values:

$$f(3) = \frac{3}{5(3)} - \frac{2(3)}{5} - 2 = \frac{3}{15} - \frac{6}{5} - 2 = \frac{1}{5} - \frac{6}{5} - 2 = -1 - 2 = -3.$$

Differentiate $f(x)$ to find $f'(x)$:

$$f'(x) = \frac{d}{dx}(\frac{3}{5}x^{-1} - \frac{2}{5}x - 2) = -\frac{3}{5}x^{-2} - \frac{2}{5} = -\frac{3}{5x^2} - \frac{2}{5}.$$

$$f'(1/4) = -\frac{3}{5(1/4)^2} - \frac{2}{5} = -\frac{3}{5(1/16)} - \frac{2}{5} = -\frac{48}{5} - \frac{2}{5} = -\frac{50}{5} = -10.$$

Finally, calculate the result:

$$|f(3) + f'(1/4)| = |-3 + (-10)| = |-13| = 13.$$

10. Suppose that a function $f : R \to R$ satisfies $f(x+y) = f(x)f(y)$ for all x, $y \in R$ and $f(1) = 3$. If $\Sigma_{i=1}^{n} f(i) = 363$, then n is equal to

Answer: 5
Solution:

The functional equation $f(x+y) = f(x)f(y)$ implies $f(x) = a^x$ for some constant a.

We are given $f(1) = 3$.

$$f(1) = a^1 = 3 \implies a = 3.$$

Therefore, the function is $f(x) = 3^x$.

We are given the sum:

$$\sum_{i=1}^{n} f(i) = \sum_{i=1}^{n} 3^i = 3^1 + 3^2 + 3^3 + \cdots + 3^n = 363.$$

This is a Geometric Progression (G.P.) with first term $a = 3$, common ratio $r = 3$, and n terms.

The sum of a G.P. is $S_n = \frac{a(r^n - 1)}{r - 1}$.

$$\frac{3(3^n - 1)}{3 - 1} = 363$$

$$\frac{3(3^n - 1)}{2} = 363$$

$$3^n - 1 = \frac{363 \times 2}{3}$$

$$3^n - 1 = 121 \times 2$$

$$3^n - 1 = 242$$

$$3^n = 243.$$
$$\text{Since } 3^5 = 243, \text{ we have } n = 5.$$

11. In a survey of 220 students of a higher secondary school, it was found that at least 125 and at most 130 students studied Mathematics; at least 85 and at most 95 studied Physics; at least 75 and at most 90 studied Chemistry; 30 studied both Physics and Chemistry; 50 studied both Chemistry and Mathematics; 40 studied both Mathematics and Physics and 10 studied none of these subjects. Let m and n respectively be the least and the most number of students who studied all the three subjects. Then $m + n$ is equal to

Answer: 45
Solution:

$$\text{Let } n(M), n(P), n(C) \text{ be the number of students studying Maths, Physics, Chemistry.}$$
$$n(U) = 220.\ n(\text{None}) = 10.$$
$$\text{So, } n(M \cup P \cup C) = n(U) - n(\text{None}) = 220 - 10 = 210.$$
$$\text{We are given:}$$
$$125 \le n(M) \le 130 \quad 85 \le n(P) \le 95 \quad 75 \le n(C) \le 90$$
$$n(P \cap C) = 30 \quad n(C \cap M) = 50 \quad n(M \cap P) = 40$$
$$\text{Let } n(M \cap P \cap C) = x.$$

$$\text{From the Principle of Inclusion-Exclusion:}$$
$$n(M \cup P \cup C) = n(M) + n(P) + n(C) - (n(M \cap P) + n(P \cap C) + n(M \cap C)) + n(M \cap P \cap C)$$
$$210 = n(M) + n(P) + n(C) - (40 + 30 + 50) + x$$
$$210 = n(M) + n(P) + n(C) - 120 + x$$
$$n(M) + n(P) + n(C) = 330 - x.$$

$$\text{1. Find } n = \max(x):$$
$$\text{The number of students in any region of the Venn diagram must be non-negative.}$$
$$n(M \cap P \text{ only}) = n(M \cap P) - x = 40 - x \ge 0 \implies x \le 40$$
$$n(P \cap C \text{ only}) = n(P \cap C) - x = 30 - x \ge 0 \implies x \le 30$$
$$n(M \cap C \text{ only}) = n(M \cap C) - x = 50 - x \ge 0 \implies x \le 50$$
$$\text{To satisfy all three, } x \text{ must be } \le 30. \text{ So, } n = \max(x) = 30.$$

$$\text{2. Find } m = \min(x):$$
$$\text{We use the range of the sums.}$$
$$\text{Max sum: } n(M) + n(P) + n(C) \le 130 + 95 + 90 = 315.$$
$$330 - x \le 315 \implies 15 \le x.$$
$$\text{Min sum: } n(M) + n(P) + n(C) \ge 125 + 85 + 75 = 285.$$
$$330 - x \ge 285 \implies 45 \ge x. \text{ (This is consistent with } n = 30).$$
$$\text{Also, } n(C \text{ only}) = n(C) - n(M \cap C \text{ only}) - n(P \cap C \text{ only}) - x \ge 0$$
$$n(C \text{ only}) = n(C) - (50 - x) - (30 - x) - x = n(C) - 80 + x \ge 0.$$
$$\text{Using min } n(C) = 75: \quad 75 - 80 + x \ge 0 \implies x \ge 5.$$
$$\text{The condition } x \ge 15 \text{ is stricter. So, } m = \min(x) = 15.$$

$$\text{Result: } m = 15, n = 30.\ m + n = 15 + 30 = 45.$$

12. The sum of all the elements of the set $\{\alpha \in \{1, 2, ...100\} : HCF(\alpha, 24) = 1\}$ is

Answer: 1633
Solution:

We want the sum of all $\alpha \in \{1, \ldots, 100\}$ such that $\gcd(\alpha, 24) = 1$.

$24 = 2^3 \cdot 3$.

This means we want α such that α is not divisible by 2 AND α is not divisible by 3.

Let S be the sum of all numbers from 1 to 100.

Let S_2 be the sum of numbers divisible by 2.

Let S_3 be the sum of numbers divisible by 3.

Let S_6 be the sum of numbers divisible by 6 (by 2 and 3).

By the Principle of Inclusion-Exclusion for sums, the required sum is $S_{req} = S - S_2 - S_3 + S_6$.

$$S = \sum_{k=1}^{100} k = \frac{100(101)}{2} = 5050.$$

$$S_2 = \sum_{k=1}^{50} 2k = 2\sum_{k=1}^{50} k = 2 \cdot \frac{50(51)}{2} = 2550.$$

$$S_3 = \sum_{k=1}^{33} 3k = 3\sum_{k=1}^{33} k = 3 \cdot \frac{33(34)}{2} = 3 \cdot 561 = 1683.$$

$$S_6 = \sum_{k=1}^{16} 6k = 6\sum_{k=1}^{16} k = 6 \cdot \frac{16(17)}{2} = 6 \cdot 136 = 816.$$

$$\begin{aligned} S_{req} &= 5050 - 2550 - 1683 + 816 \\ &= 2500 - 1683 + 816 \\ &= 817 + 816 = 1633. \end{aligned}$$

13. Let $A = \{1, 2, 3, \ldots.100\}$. Let R be a relation on A defined by $(x, y) \in R$ if and only if $2x = 3y$. Let R_1 be a symmetric relation on A such that $R \subset R_1$ and the number of elements in R_1 is n. Then the minimum value of n is

Answer: 66
Solution:

$R = \{(x, y) \in A \times A \mid 2x = 3y\}$.

This implies x must be a multiple of 3 and y must be a multiple of 2.

Let $x = 3k$ and $y = 2k$.

We need $x \in A \implies 1 \leq 3k \leq 100 \implies k \leq 33$.

We need $y \in A \implies 1 \leq 2k \leq 100 \implies k \leq 50$.

Both conditions require $k \in \{1, 2, \ldots, 33\}$.

So, $R = \{(3, 2), (6, 4), (9, 6), \ldots, (99, 66)\}$.

The number of elements in R is $n(R) = 33$.

R_1 is the smallest symmetric relation containing R.

This means $R_1 = R \cup R^{-1}$.

$R^{-1} = \{(y, x) \mid (x, y) \in R\} = \{(2, 3), (4, 6), (6, 9), \ldots, (66, 99)\}$.

$n(R^{-1}) = 33$.

The number of elements in R_1 is $n(R_1) = n(R \cup R^{-1}) = n(R) + n(R^{-1}) - n(R \cap R^{-1})$.

An element (x, y) is in $R \cap R^{-1}$ if $(x, y) \in R$ and $(y, x) \in R$.

$(x, y) \in R \implies 2x = 3y$.

$$(y, x) \in R \implies 2y = 3x.$$
Substituting $x = 3y/2$ into the 2nd equation: $2y = 3(3y/2)$
$$\implies 2y = 9y/2 \implies 4y = 9y \implies 5y = 0 \implies y = 0.$$
If $y = 0, x = 0$. But $(0, 0) \notin A \times A$.
Therefore, $R \cap R^{-1} = \emptyset$ (the empty set).

$$n = n(R_1) = n(R) + n(R^{-1}) - 0 = 33 + 33 = 66.$$

14. Let Z be the set of integers. If $A = \{x \in Z : 2^{(x+2)(x^2-5x+6)} = 1\}$ and $B = \{x \in Z : -3 < 2x - 1 < 9\}$, then the number of subsets of the set $A \times B$ is:

(A) 2^{12} (B) 2^{10} (C) 2^{18} (D) 2^{15}

Answer: (D)
Solution:

1. Find Set A:
$$2^{(x+2)(x^2-5x+6)} = 1 = 2^0.$$
This implies the exponent must be zero.
$$(x + 2)(x^2 - 5x + 6) = 0$$
$$(x + 2)(x - 2)(x - 3) = 0.$$
Since $x \in \mathbb{Z}$, the solutions are $x = -2, 2, 3$.
$A = \{-2, 2, 3\}$. So, $n(A) = 3$.

2. Find Set B:
$$-3 < 2x - 1 < 9.$$
Add 1 to all parts:
$$-2 < 2x < 10.$$
Divide by 2:
$$-1 < x < 5.$$
Since $x \in \mathbb{Z}$, the solutions are $x = 0, 1, 2, 3, 4$.
$B = \{0, 1, 2, 3, 4\}$. So, $n(B) = 5$.

3. Find $n(A \times B)$:
The number of elements in the Cartesian product $A \times B$ is:
$$n(A \times B) = n(A) \times n(B) = 3 \times 5 = 15.$$

4. Find the number of subsets:
The number of subsets of a set with k elements is 2^k.
Number of subsets of $A \times B = 2^{n(A \times B)} = 2^{15}$.

15. Let a relation R on $N \times N$ be defined as: $(x_1, y_1)R(x_2, y_2)$ if and only if $x_1 \leq x_2$ or $y_1 \leq y_2$. Consider the two statements: (I) R is reflexive but not symmetric. (II) R is transitive. Then which one of the following is true?

(A) Both (I) and (II) are correct. (B) Only (II) is correct.
(C) Neither (I) nor (II) is correct. (D) Only (I) is correct.

Answer: (D)
Solution:

$$\text{Let } a = (x_1, y_1), b = (x_2, y_2), c = (x_3, y_3).$$
$$\text{The relation is } aRb \iff x_1 \le x_2 \text{ or } y_1 \le y_2.$$

$$\text{(I) Reflexive and Symmetric:}$$
$$\text{Reflexive: Is } aRa\text{? i.e., } (x_1, y_1)R(x_1, y_1).$$
$$\text{This requires } x_1 \le x_1 \text{ or } y_1 \le y_1. \text{ Since } x_1 = x_1, \text{ this is always true. } \textbf{R is reflexive.}$$
$$\text{Symmetric: If } aRb, \text{ is } bRa?$$
$$\text{Let } a = (5, 5) \text{ and } b = (1, 1).$$
$$bRa: \text{ Is } (1, 1)R(5, 5)\text{? Yes, because } 1 \le 5. \text{ So } bRa \text{ is True.}$$
$$aRb: \text{ Is } (5, 5)R(1, 1)\text{? No, because } 5 \not\le 1 \text{ AND } 5 \not\le 1. \text{ So } aRb \text{ is False.}$$
$$\text{We found a case where } bRa \text{ is true but } aRb \text{ is false. } \textbf{R is not symmetric.}$$
$$\text{Therefore, statement (I) is correct.}$$

$$\text{(II) Transitive:}$$
$$\text{If } aRb \text{ and } bRc, \text{ is } aRc?$$
$$\text{Let's try to find a counterexample.}$$
$$\text{Let } a = (2, 5),\ b = (3, 3),\ c = (1, 4).$$
$$aRb: \text{ Is } (2, 5)R(3, 3)\text{? Yes, because } 2 \le 3. \text{ (True)}$$
$$bRc: \text{ Is } (3, 3)R(1, 4)\text{? Yes, because } 3 \le 4. \text{ (True)}$$
$$aRc: \text{ Is } (2, 5)R(1, 4)\text{? No, because } 2 \not\le 1 \text{ AND } 5 \not\le 4. \text{ (False)}$$
$$\text{We found a case where } aRb \text{ and } bRc \text{ but not } aRc.$$
$$\textbf{R is not transitive.} \text{ Therefore, statement (II) is incorrect.}$$

$$\text{Only statement (I) is correct.}$$

16. Let $A = \{2, 3, 6, 8, 9, 11\}$ and $B = \{1, 4, 5, 10, 15\}$. Let R be a relation on $A \times B$ defined by $(a, b)R(c, d)$ if and only if $3ad - 7bc$ is an even integer. Then the relation R is

(A) an equivalence relation.
(B) reflexive and symmetric but not transitive.
(C) transitive but not symmetric.
(D) reflexive but not symmetric.

Answer: (B)
Solution:

$$\text{Let } E_1 = (a, b), E_2 = (c, d), E_3 = (e, f).$$
$$\text{The condition } 3ad - 7bc \text{ is even means } 3ad - 7bc \equiv 0 \pmod 2.$$
$$\text{Since } 3 \equiv 1 \pmod 2 \text{ and } 7 \equiv 1 \pmod 2, \text{ this simplifies to:}$$
$$ad - bc \equiv 0 \pmod 2 \text{ or } \mathbf{ad \equiv bc \pmod 2}.$$
$$\text{(i.e., } ad \text{ and } bc \text{ have the same parity).}$$

$$\text{Reflexive: Is } E_1 R E_1\text{? i.e., } (a, b)R(a, b).$$
$$\text{Condition: } ab \equiv ba \pmod 2. \text{ This is always true. } \textbf{R is reflexive.}$$

$$\text{Symmetric: If } E_1 R E_2, \text{ is } E_2 R E_1?$$
$$\text{Given: } (a, b)R(c, d) \implies ad \equiv bc \pmod 2.$$
$$\text{Check: } (c, d)R(a, b) \implies cb \equiv da \pmod 2.$$
$$\text{Since } ad \equiv bc \text{ is the same as } cb \equiv da, \text{ this is always true. } \textbf{R is symmetric.}$$

$$\text{Transitive: If } E_1RE_2 \text{ and } E_2RE_3, \text{ is } E_1RE_3?$$
$$\text{Given: (1)} \quad ad \equiv bc \pmod 2$$
$$\text{Given: (2)} \quad cf \equiv de \pmod 2$$
$$\text{Check: (3)} \quad af \equiv be \pmod 2?$$
Let's find a counterexample. We need to break the chain of parity.
$$\text{Let } E_1 = (a,b) = (3,1). \quad (a \text{ is Odd, } b \text{ is Odd})$$
$$\text{Let } E_2 = (c,d) = (6,4). \quad (c \text{ is Even, } d \text{ is Even})$$
$$\text{Let } E_3 = (e,f) = (2,5). \quad (e \text{ is Even, } f \text{ is Odd})$$
All elements are from A and B.
$$\text{Check } E_1RE_2: \quad ad \equiv bc \pmod 2$$
$$3 \cdot 4 \equiv 1 \cdot 6 \pmod 2 \implies 12 \equiv 6 \pmod 2 \implies 0 \equiv 0. \text{ (True)}$$
$$\text{Check } E_2RE_3: \quad cf \equiv de \pmod 2$$
$$6 \cdot 5 \equiv 4 \cdot 2 \pmod 2 \implies 30 \equiv 8 \pmod 2 \implies 0 \equiv 0. \text{ (True)}$$
$$\text{Check } E_1RE_3: \quad af \equiv be \pmod 2$$
$$3 \cdot 5 \equiv 1 \cdot 2 \pmod 2 \implies 15 \equiv 2 \pmod 2 \implies 1 \equiv 0. \text{ (False)}$$
We found a case where E_1RE_2 and E_2RE_3 but not E_1RE_3.

R is not transitive.

The relation is reflexive and symmetric but not transitive.

17. Let $A = \{1,3,4,6,9\}$ and $B = \{2,4,5,8,10\}$. Let R be a relation defined on $A \times B$ such that $R = \{(a_1,b_1),(a_2,b_2) : a_1 \le b_2 \text{ and } b_1 \le a_2\}$ Then the number of elements in the set R is

(A) 160 (B) 52 (C) 26 (D) 180

Answer: (A)
Solution:

The relation R is on the set $S = A \times B$. So $R \subset S \times S$.

An element of R is a pair of elements from S.

Let $E_1 = (a_1,b_1) \in S$ and $E_2 = (a_2,b_2) \in S$.
$$(E_1,E_2) \in R \iff a_1 \le b_2 \text{ and } b_1 \le a_2.$$

We want to find $n(R)$, which is the number of such pairs (E_1,E_2).
$$n(R) = \sum_{E_1 \in S} \sum_{E_2 \in S} \mathbb{I}(a_1 \le b_2 \text{ and } b_1 \le a_2)$$
where $\mathbb{I}$ is the indicator function (1 if true, 0 if false).
$$n(R) = \sum_{(a_1,b_1)\in A\times B} \left(\sum_{(a_2,b_2)\in A\times B} \mathbb{I}(a_1 \le b_2) \cdot \mathbb{I}(b_1 \le a_2) \right)$$
The inner sum is separable because a_2 and b_2 are independent.
$$n(R) = \sum_{(a_1,b_1)\in A\times B} \left(\sum_{a_2 \in A} \mathbb{I}(b_1 \le a_2) \right) \cdot \left(\sum_{b_2 \in B} \mathbb{I}(a_1 \le b_2) \right)$$
This outer sum is also separable.
$$n(R) = \left(\sum_{a_1 \in A} \sum_{b_2 \in B} \mathbb{I}(a_1 \le b_2) \right) \cdot \left(\sum_{b_1 \in B} \sum_{a_2 \in A} \mathbb{I}(b_1 \le a_2) \right)$$

$$\text{Let } N_1 = \sum_{a\in A} \sum_{b \in B} \mathbb{I}(a \le b). \text{ (Number of pairs } (a,b) \text{ with } a \le b)$$

$$\text{Let } N_2 = \sum_{b\in B}\sum_{a\in A} \mathbb{I}(b \le a). \text{ (Number of pairs } (b,a) \text{ with } b \le a)$$

$$A = \{1,3,4,6,9\}, B = \{2,4,5,8,10\}$$

$$N_1: \quad a = 1 \implies b \in \{2,4,5,8,10\} \text{ (5 pairs)}$$
$$a = 3 \implies b \in \{4,5,8,10\} \text{ (4 pairs)}$$
$$a = 4 \implies b \in \{4,5,8,10\} \text{ (4 pairs)}$$
$$a = 6 \implies b \in \{8,10\} \text{ (2 pairs)}$$
$$a = 9 \implies b \in \{10\} \text{ (1 pair)}$$
$$N_1 = 5+4+4+2+1 = 16.$$

$$N_2: \quad b = 2 \implies a \in \{3,4,6,9\} \text{ (4 pairs)}$$
$$b = 4 \implies a \in \{4,6,9\} \text{ (3 pairs)}$$
$$b = 5 \implies a \in \{6,9\} \text{ (2 pairs)}$$
$$b = 8 \implies a \in \{9\} \text{ (1 pair)}$$
$$b = 10 \implies a \in \{\} \text{ (0 pairs)}$$
$$N_2 = 4+3+2+1+0 = 10.$$

$$n(R) = N_1 \times N_2 = 16 \times 10 = 160.$$

18. Let $A = \{1, 2, 3, 4, 5\}$. Let R be a relation on A defined by xRy if and only if $4x \le 5y$. Let m be the number of elements in R, and n be the minimum number of elements from $A \times A$ that are required to be added to R to make it a symmetric relation. Then $m + n$ is equal to:

(A) 25 (B) 24 (C) 26 (D) 23

Answer: (A)
Solution:

$$A = \{1,2,3,4,5\}. \ R = \{(x,y) \in A \times A \mid 4x \le 5y\}.$$

$$\text{1. Find } m = n(R):$$
$$\text{We list the pairs } (x,y) \text{ in R by checking each } x:$$
$$x = 1: 4 \le 5y \implies y \in \{1,2,3,4,5\} \text{ (5 pairs)}$$
$$x = 2: 8 \le 5y \implies y \in \{2,3,4,5\} \text{ (4 pairs)}$$
$$x = 3: 12 \le 5y \implies y \in \{3,4,5\} \text{ (3 pairs)}$$
$$x = 4: 16 \le 5y \implies y \in \{4,5\} \text{ (2 pairs)}$$
$$x = 5: 20 \le 5y \implies y \in \{4,5\} \text{ (2 pairs)}$$
$$m = n(R) = 5+4+3+2+2 = 16.$$

$$\text{2. Find } n = \text{ elements to add for symmetry}:$$
$$\text{A symmetric relation must contain } R \cup R^{-1}.$$
$$n \text{ is the number of elements in } R^{-1} \text{ that are not in } R.$$
$$n = n(R^{-1} - R) = n(R^{-1}) - n(R \cap R^{-1}).$$

$$\text{First, find } n(R^{-1}) \text{ (elements } (x,y) \text{ s.t. } 4y \le 5x):$$
$$y = 1: 4 \le 5x \implies x \in \{1,2,3,4,5\} \text{ (5 pairs)}$$
$$y = 2: 8 \le 5x \implies x \in \{2,3,4,5\} \text{ (4 pairs)}$$
$$y = 3: 12 \le 5x \implies x \in \{3,4,5\} \text{ (3 pairs)}$$
$$y = 4: 16 \le 5x \implies x \in \{4,5\} \text{ (2 pairs)}$$
$$y = 5: 20 \le 5x \implies x \in \{4,5\} \text{ (2 pairs)}$$

$$n(R^{-1}) = 5 + 4 + 3 + 2 + 2 = 16.$$

Next, find $n(R \cap R^{-1})$ (elements (x, y) s.t. $4x \le 5y$ AND $4y \le 5x$) :

All diagonal elements $(x, x) : 4x \le 5x$ (True). So $(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)$ are in. (5 pairs)

Off-diagonal $(x \ne y)$:

$(3, 4) : 4(3) \le 5(4)$ (T), $4(4) \le 5(3)$ (F). No.

$(4, 5) : 4(4) \le 5(5)$ (T), $4(5) \le 5(4)$ (T). Yes, (4,5) is in.

$(5, 4) : 4(5) \le 5(4)$ (T), $4(4) \le 5(5)$ (T). Yes, (5,4) is in.

$$n(R \cap R^{-1}) = 5 + 2 = 7.$$

$$n = n(R^{-1}) - n(R \cap R^{-1}) = 16 - 7 = 9.$$

$$m + n = 16 + 9 = 25.$$

19. The minimum number of elements that must be added to the relation $R = \{(a, b), (b, c)\}$ on the set {a, b, c} so that it becomes symmetric and transitive is:

(A) 4 (B) 7 (C) 5 (D) 3

Answer: (B)
Solution:

Let $S = \{a, b, c\}$. The initial relation is $R_0 = \{(a, b), (b, c)\}$. $n(R_0) = 2$.

We need to add elements to make it symmetric and transitive.

Note: The problem does not require it to be reflexive.

However, symmetric + transitive implies $(a, b) \in R \implies (b, a) \in R$, and $(a, b), (b, a) \in R \implies (a, a) \in R$.

So, it must also be reflexive on the elements involved (a,b,c).

1. Add for Symmetry:

We have (a, b), so we must add (b, a).

We have (b, c), so we must add (c, b).

$R_1 = \{(a, b), (b, c), (b, a), (c, b)\}$.

2. Add for Transitivity (using R_1) :

(a, b) and $(b, a) \implies$ add (a, a).

(b, c) and $(c, b) \implies$ add (b, b).

(c, b) and $(b, c) \implies$ add (c, c).

(a, b) and $(b, c) \implies$ add (a, c).

(c, b) and $(b, a) \implies$ add (c, a).

$R_2 = \{(a, b), (b, c), (b, a), (c, b), (a, a), (b, b), (c, c), (a, c), (c, a)\}$.

3. Check for further additions:

The set R_2 is $S \times S$, which is the universal relation. It is symmetric and transitive.

The final set is R_2, which has $3 \times 3 = 9$ elements.

The original set R_0 had 2 elements.

Number of elements added $= n(R_2) - n(R_0) = 9 - 2 = 7$.

20. If R is the smallest equivalence relation on the set $\{1, 2, 3, 4\}$ such that $\{(1, 2), (1, 3)\} \subset R$, then the number of elements in R is

(A) 10 (B) 12 (C) 8 (D) 15

Answer: (A)
Solution:

$$\text{Let } S = \{1, 2, 3, 4\}. \ R \text{ must contain } \{(1, 2), (1, 3)\}.$$
$$\text{An equivalence relation must be reflexive, symmetric, and transitive.}$$

$$\text{1. For Reflexivity:}$$
$$R \text{ must contain } \{(1, 1), (2, 2), (3, 3), (4, 4)\}.$$

$$\text{2. For Symmetry:}$$
$$\text{Since } (1, 2) \in R, \text{ we must add } (2, 1).$$
$$\text{Since } (1, 3) \in R, \text{ we must add } (3, 1).$$
$$R \text{ now contains } \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (1, 3), (2, 1), (3, 1)\}.$$

$$\text{3. For Transitivity:}$$
$$\text{We check all combinations. Let's trace the connections:}$$
$$(2, 1) \in R \text{ and } (1, 3) \in R \implies \text{we must add } (2, 3).$$
$$(3, 1) \in R \text{ and } (1, 2) \in R \implies \text{we must add } (3, 2).$$

$$\text{Now } R \text{ contains } \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)\}.$$

$$\text{Let's check again:}$$
$$(1, 2) \in R \text{ and } (2, 3) \in R \implies (1, 3) \in R. \text{ (Have)}$$
$$(3, 2) \in R \text{ and } (2, 1) \in R \implies (3, 1) \in R. \text{ (Have)}$$
$$\text{All other combinations are satisfied.}$$

$$\text{The elements } \{1, 2, 3\} \text{ are all related to each other, forming an equivalence class.}$$
$$\text{The element } \{4\} \text{ is only related to itself.}$$
$$\text{The final relation } R \text{ is } (\{1, 2, 3\} \times \{1, 2, 3\}) \cup \{(4, 4)\}.$$
$$\text{Number of elements} = (3 \times 3) + 1 = 9 + 1 = 10.$$

21. A survey shows that 73

(A) 63 (B) 36 (C) 54 (D) 38

Answer: (B)
Solution:

$$\text{Let } n(U) = 100.$$
$$\text{Let } C \text{ be the set of people who like coffee, } n(C) = 73.$$
$$\text{Let } T \text{ be the set of people who like tea, } n(T) = 65.$$
$$\text{Let } x = n(C \cap T).$$

$$\text{1. Find the maximum value of } x:$$
$$\text{The intersection cannot be larger than the smaller of the two sets.}$$
$$x \leq \min(n(C), n(T)) \implies x \leq \min(73, 65) \implies x \leq 65.$$

2. Find the minimum value of x :

The union of the two sets cannot be larger than the universal set.

$$n(C \cup T) \leq n(U)$$
$$n(C) + n(T) - n(C \cap T) \leq 100$$
$$73 + 65 - x \leq 100$$
$$138 - x \leq 100$$
$$138 - 100 \leq x \implies x \geq 38.$$

So, the range for x is $38 \leq x \leq 65$.

From the options, 36 is not in this interval.

Therefore, x cannot be 36.

22. In a class of 140 students numbered 1 to 140, all even numbered students opted Mathematics course, those whose number is divisible by 3 opted Physics course and those whose number is divisible by 5 opted Chemistry course. Then the number of students who did not opt for any of the three courses is:

(A) 42 (B) 102 (C) 38 (D) 1

Answer: (C)
Solution:

Let $N = 140$.

Let M be the set of students who opted for Maths (divisible by 2).

Let P be the set of students who opted for Physics (divisible by 3).

Let C be the set of students who opted for Chemistry (divisible by 5).

We want to find $N - n(M \cup P \cup C)$.

We use the Principle of Inclusion-Exclusion,

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - (n(A \cap B) + \dots) + n(A \cap B \cap C).$$

$$n(M) = \lfloor 140/2 \rfloor = 70.$$
$$n(P) = \lfloor 140/3 \rfloor = 46.$$
$$n(C) = \lfloor 140/5 \rfloor = 28.$$

$$n(M \cap P) = \lfloor 140/\text{LCM}(2,3) \rfloor = \lfloor 140/6 \rfloor = 23.$$
$$n(P \cap C) = \lfloor 140/\text{LCM}(3,5) \rfloor = \lfloor 140/15 \rfloor = 9.$$
$$n(M \cap C) = \lfloor 140/\text{LCM}(2,5) \rfloor = \lfloor 140/10 \rfloor = 14.$$

$$n(M \cap P \cap C) = \lfloor 140/\text{LCM}(2,3,5) \rfloor = \lfloor 140/30 \rfloor = 4.$$

$$\begin{aligned} n(M \cup P \cup C) &= (n(M) + n(P) + n(C)) - (n(M \cap P) + n(P \cap C) + n(M \cap C)) + n(M \cap P \cap C) \\ &= (70 + 46 + 28) - (23 + 9 + 14) + 4 \\ &= 144 - 46 + 4 \\ &= 98 + 4 = 102. \end{aligned}$$

$$\begin{aligned} \text{Number of students who did not opt for any course} &= N - n(M \cup P \cup C) \\ &= 140 - 102 = 38. \end{aligned}$$

23. Let $A = \{1, 2, 3, ..., 10\}$ and $B = \{\frac{m}{n} : m, n \in A$, m ¡ n and $gcd(m, n) = 1\}$. Then $n(B)$ is equal to:

(A) 36 (B) 31 (C) 37 (D) 29

Answer: (B)
Solution:

$$\text{The set } B \text{ contains unique fractions } m/n \text{ where } m, n \in \{1, \ldots, 10\}, m < n, \text{ and } \gcd(m, n) = 1.$$
$$\text{The number of elements in } B, n(B), \text{ is the count of such pairs } (m, n).$$
$$\text{This is equivalent to finding the number of coprime integers } m \text{ for each } n \text{ from 2 to 10,}$$
$$\text{such that } 1 \leq m < n. \text{ This is the definition of Euler's Totient Function, } \phi(n).$$
$$n(B) = \sum_{n=2}^{10} \phi(n).$$

$$\phi(2) = 1 \quad (\text{pairs: } (1, 2))$$
$$\phi(3) = 2 \quad (\text{pairs: } (1, 3), (2, 3))$$
$$\phi(4) = 2 \quad (\text{pairs: } (1, 4), (3, 4))$$
$$\phi(5) = 4 \quad (\text{pairs: } (1, 5), (2, 5), (3, 5), (4, 5))$$
$$\phi(6) = 2 \quad (\text{pairs: } (1, 6), (5, 6))$$
$$\phi(7) = 6 \quad (\text{pairs: } (1, 7), \ldots, (6, 7))$$
$$\phi(8) = 4 \quad (\text{pairs: } (1, 8), (3, 8), (5, 8), (7, 8))$$
$$\phi(9) = 6 \quad (\text{pairs: } (1, 9), (2, 9), (4, 9), (5, 9), (7, 9), (8, 9))$$
$$\phi(10) = 4 \quad (\text{pairs: } (1, 10), (3, 10), (7, 10), (9, 10))$$

$$n(B) = 1 + 2 + 2 + 4 + 2 + 6 + 4 + 6 + 4 = 31.$$

24. If the range of the function $f(x) = \frac{5-x}{x^2-3x+2}, x \neq 1, 2,$ is $(-\infty, \alpha] \cup [\beta, \infty)$, then $\alpha^2 + \beta^2$ is equal to:

(A) 190 (B) 192 (C) 188 (D) 194

Answer: (D)
Solution:

$$\text{Let } y = \frac{5 - x}{x^2 - 3x + 2}.$$
$$y(x^2 - 3x + 2) = 5 - x$$
$$yx^2 - 3yx + 2y = 5 - x$$
$$yx^2 + (1 - 3y)x + (2y - 5) = 0.$$
$$\text{For } x \text{ to be real, the discriminant } D \text{ must be } \geq 0.$$
$$D = (1 - 3y)^2 - 4(y)(2y - 5) \geq 0$$
$$(1 - 6y + 9y^2) - (8y^2 - 20y) \geq 0$$
$$y^2 + 14y + 1 \geq 0.$$

$$\text{The roots of the quadratic } y^2 + 14y + 1 = 0 \text{ are given by the quadratic formula:}$$
$$y = \frac{-14 \pm \sqrt{14^2 - 4(1)(1)}}{2} = \frac{-14 \pm \sqrt{196 - 4}}{2}$$
$$y = \frac{-14 \pm \sqrt{192}}{2} = \frac{-14 \pm \sqrt{64 \times 3}}{2} = \frac{-14 \pm 8\sqrt{3}}{2}$$
$$y = -7 \pm 4\sqrt{3}.$$

$$\text{Since } y^2 + 14y + 1 \geq 0, \text{ the range is outside the roots.}$$
$$\text{Range} = (-\infty, -7 - 4\sqrt{3}] \cup [-7 + 4\sqrt{3}, \infty).$$

$$\text{So, } \alpha = -7 - 4\sqrt{3} \text{ and } \beta = -7 + 4\sqrt{3}.$$
$$\text{We need to find } \alpha^2 + \beta^2.$$
$$\text{We can use the identity } \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta.$$
$$\text{From the quadratic } y^2 + 14y + 1 = 0 :$$
$$\text{Sum of roots } \alpha + \beta = -14.$$
$$\text{Product of roots } \alpha\beta = 1.$$

$$\alpha^2 + \beta^2 = (-14)^2 - 2(1) = 196 - 2 = 194.$$

25. Let $f : (1, 3) \to R$ be a function defined by $f(x) = \frac{x[x]}{1+x^2}$, where $[x]$ denotes the greatest integer $\leq x$. Then the range of f is

(A) $(\frac{2}{5}, \frac{3}{5}] \cup (\frac{3}{4}, \frac{4}{5})$
(B) $(\frac{2}{5}, \frac{1}{2}) \cup (\frac{3}{5}, \frac{4}{5}]$
(C) $(\frac{2}{5}, \frac{4}{5}]$
(D) $(\frac{3}{5}, \frac{4}{5}]$

Answer: (B)
Solution:

$$\text{The domain is } (1, 3). \text{ We must split this interval based on the value of } [x].$$

$$\text{Case 1: } x \in (1, 2)$$
$$\text{In this interval, } [x] = 1.$$
$$f(x) = \frac{x(1)}{1 + x^2} = \frac{x}{1 + x^2}.$$
$$\text{To find the range, let's check the derivative:}$$
$$f'(x) = \frac{(1)(1 + x^2) - x(2x)}{(1 + x^2)^2} = \frac{1 - x^2}{(1 + x^2)^2}.$$
$$\text{For } x \in (1, 2), \ 1 - x^2 < 0, \text{ so } f'(x) < 0. \text{ The function is decreasing.}$$
$$\text{The range is } (f(2), f(1)) = (\frac{2}{1+4}, \frac{1}{1+1}) = (\frac{2}{5}, \frac{1}{2}).$$

$$\text{Case 2: } x \in [2, 3)$$
$$\text{In this interval, } [x] = 2.$$
$$f(x) = \frac{x(2)}{1 + x^2} = \frac{2x}{1 + x^2}.$$
$$\text{Let's check the derivative:}$$
$$f'(x) = \frac{(2)(1 + x^2) - 2x(2x)}{(1 + x^2)^2} = \frac{2 + 2x^2 - 4x^2}{(1 + x^2)^2} = \frac{2 - 2x^2}{(1 + x^2)^2}.$$
$$\text{For } x \in [2, 3), \ 2 - 2x^2 < 0, \text{ so } f'(x) < 0. \text{ The function is decreasing.}$$
$$\text{The range is } (f(3), f(2)] = (\frac{2(3)}{1+9}, \frac{2(2)}{1+4}] = (\frac{6}{10}, \frac{4}{5}] = (\frac{3}{5}, \frac{4}{5}].$$

$$\text{Total Range:}$$
$$\text{The total range is the union of the ranges from Case 1 and Case 2.}$$
$$\text{Range} = (\frac{2}{5}, \frac{1}{2}) \cup (\frac{3}{5}, \frac{4}{5}].$$