

3. INDEFINITE INTEGRATION



Let us Study

- Definition and Properties
- Different Techniques : 1. by substitution 2. by parts 3. by partial fraction

Introduction :

In differential calculus, we studied differentiation or derivatives of some functions. We saw that derivatives are used for finding the slopes of tangents, maximum or minimum values of the function.

Now we will try to find the function whose derivative is known, or given $f(x)$. We will find $g(x)$ such that $g'(x) = f(x)$. Here the integration of $f(x)$ with respect to x is $g(x)$ or $g(x)$ is called the primitive of $f(x)$. For example, we know that the derivative of x^3 w. r. t. x is $3x^2$. So $\frac{d}{dx} x^3 = 3x^2$; and integral of $3x^2$ w. r. t. x is x^3 . This is shown with the sign of integration namely ' $\int$ '. We write $\int 3x^2 dx = x^3$.

In this chapter we restrict ourselves only to study the methods of integration. The theory of integration is developed by Sir Isaac Newton and Gottfried Leibnitz.

$\int f(x) dx = g(x)$, read as an integral of $f(x)$ with respect to x , is $g(x)$. Since the derivative of constant function with respect to x is zero (0), we can also write

$\int f(x) dx = g(x) + c$, where c is an arbitrary constant and c can take infinitely many values.

For example :

$f(x) = x^2 + c$ represents family of curves for different values of c .

$f'(x) = 2x$ gives the slope of the tangent to $f(x) = x^2 + c$.

In the figure we have shown the curves

$$y = x^2, y = x^2 + 4, y = x^2 - 5.$$

Note that at the points (2, 4), (2, 8) (2, -1) respectively on those curves, the slopes of tangents are $2(2) = 4$.

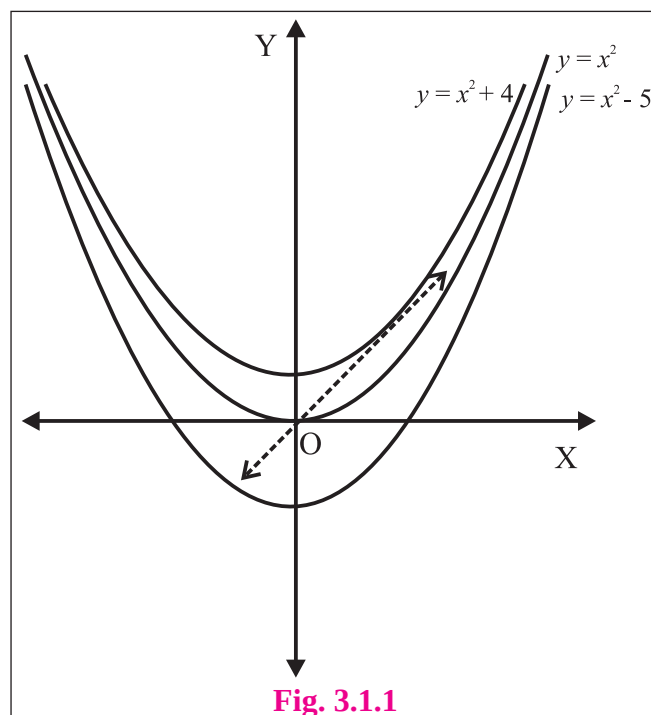


Fig. 3.1.1

3.1.1 Elementary Integration Formulae

$$\begin{aligned}
 \text{(i)} \quad \frac{d}{dx} \left(\frac{x^{n+1}}{n+1} \right) &= \frac{(n+1)x^n}{(n+1)}, n \neq -1 \\
 &= x^n \quad \Rightarrow \quad \therefore \int x^n dx = \frac{x^{n+1}}{n+1} + c \\
 \frac{d}{dx} \left(\frac{(ax+b)^{n+1}}{(n+1) \cdot a} \right) &= \frac{(n+1)(ax+b)^n}{(n+1)} \\
 &= (ax+b)^n \quad \Rightarrow \quad \therefore \int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{n+1} \cdot \frac{1}{a} + c
 \end{aligned}$$

This result can be extended for n replaced by any rational $\frac{p}{q}$.

$$\begin{aligned}
 \text{(ii)} \quad \frac{d}{dx} \left(\frac{a^x}{\log a} \right) &= a^x, a > 0 \quad \Rightarrow \quad \therefore \int a^x dx = \frac{a^x}{\log a} + c \\
 &\quad \int A^{ax+b} dx = \frac{A^{ax+b}}{\log A} \cdot \frac{1}{a} + c, A > 0 \\
 \text{(iii)} \quad \frac{d}{dx} e^x &= e^x \quad \Rightarrow \quad \int e^x dx = e^x + c \\
 &\quad \int e^{ax+b} dx = e^{ax+b} \cdot \frac{1}{a} + c \\
 \text{(iv)} \quad \frac{d}{dx} \sin x &= \cos x \quad \Rightarrow \quad \int \cos x dx = \sin x + c \\
 &\quad \int \cos(ax+b) dx = \sin(ax+b) \cdot \frac{1}{a} + c \\
 \text{(v)} \quad \frac{d}{dx} \cos x &= -\sin x \quad \Rightarrow \quad \int \sin x dx = -\cos x + c \\
 &\quad \int \sin(ax+b) dx = -\cos(ax+b) \cdot \frac{1}{a} + c \\
 \text{(vi)} \quad \frac{d}{dx} \tan x &= \sec^2 x \quad \Rightarrow \quad \int \sec^2 x dx = \tan x + c \\
 &\quad \int \sec^2(ax+b) dx = \tan(ax+b) \cdot \frac{1}{a} + c \\
 \text{(vii)} \quad \frac{d}{dx} \sec x &= \sec x \cdot \tan x \quad \Rightarrow \quad \int \sec x \cdot \tan x dx = \sec x + c \\
 &\quad \int \sec(ax+b) \cdot \tan(ax+b) dx = \sec(ax+b) \cdot \frac{1}{a} + c \\
 \text{(viii)} \quad \frac{d}{dx} \operatorname{cosec} x &= -\operatorname{cosec} x \cdot \cot x \quad \Rightarrow \quad \int \operatorname{cosec} x \cdot \cot x dx = -\operatorname{cosec} x + c \\
 &\quad \int \operatorname{cosec}(ax+b) \cdot \cot(ax+b) dx = -\operatorname{cosec}(ax+b) \cdot \frac{1}{a} + c \\
 \text{(ix)} \quad \frac{d}{dx} \cot x &= -\operatorname{cosec}^2 x \quad \Rightarrow \quad \int \operatorname{cosec}^2 x dx = -\cot x + c \\
 &\quad \int \operatorname{cosec}^2(ax+b) dx = -\cot(ax+b) \cdot \frac{1}{a} + c \\
 \text{(x)} \quad \frac{d}{dx} \log x &= \frac{1}{x}, x > 0 \quad \Rightarrow \quad \int \frac{1}{x} dx = \log x + c, x \neq 0. \\
 \therefore \text{also } \int \frac{1}{(ax+b)} dx &= \log(ax+b) \cdot \frac{1}{a} + c
 \end{aligned}$$

We assume that the trigonometric functions and logarithmic functions are defined on the respective domains.

3.1.2

Theorem 1 : If f and g are real valued integrable functions of x , then

$$\int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx$$

Theorem 2 : If f and g are real valued integrable functions of x , then

$$\int [f(x) - g(x)] dx = \int f(x) dx - \int g(x) dx$$

Theorem 3 : If f and g are real valued integrable functions of x , and k is constant, then

$$\int k [f(x)] dx = k \int f(x) dx$$

Proof : 1. Let $\int f(x) dx = g_1(x) + c_1$ and $\int g(x) dx = g_2(x) + c_2$ then

$$\begin{aligned} \frac{d}{dx} [(g_1(x) + c_1)] &= f(x) \quad \text{and} \quad \frac{d}{dx} [(g_2(x) + c_2)] = g(x) \\ \therefore \frac{d}{dx} [(g_1(x) + c_1) + (g_2(x) + c_2)] \\ &= \frac{d}{dx} [(g_1(x) + c_1)] + \frac{d}{dx} [(g_2(x) + c_2)] \\ &= f(x) + g(x) \end{aligned}$$

By definition of integration.

$$\begin{aligned} \int f(x) + g(x) dx &= (g_1(x) + c_1) + (g_2(x) + c_2) \\ &= \int f(x) dx + \int g(x) dx \end{aligned}$$

Note : Students can construct the proofs of the other two theorems (Theorem 2 and Theorem 3).



SOLVED EXAMPLES

Ex. : Evaluate the following :

1. $\int (x^3 + 3^x) dx$

Solution :

$$\begin{aligned} &\int (x^3 + 3^x) dx \\ &= \int x^3 dx + \int 3^x dx \\ &= \frac{x^4}{4} + \frac{3^x}{\log 3} + c \end{aligned}$$

$$2. \quad \int \left(\sin x + \frac{1}{x} + \frac{1}{\sqrt[3]{x}} \right) dx$$

Solution :

$$\begin{aligned} & \int \left(\sin x + \frac{1}{x} + \frac{1}{\sqrt[3]{x}} \right) dx \\ &= \int \sin x \, dx + \int \frac{1}{x} \, dx + \int \frac{1}{\sqrt[3]{x}} \, dx \\ &= \int \sin x \, dx + \int \frac{1}{x} \, dx + \int x^{-\frac{1}{3}} \, dx \\ &= -\cos x + \log x + \frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} + c \\ &= -\cos x + \log x + \frac{x^{\frac{2}{3}}}{\frac{2}{3}} + c \end{aligned}$$

$$4. \quad \int \frac{\sqrt{x} + 1}{x + \sqrt{x}} \, dx$$

Solution :

$$\begin{aligned} & \int \frac{\sqrt{x} + 1}{x + \sqrt{x}} \, dx \\ &= \int \frac{\sqrt{x} + 1}{\sqrt{x}(\sqrt{x} + 1)} \, dx \\ &= \int \frac{1}{\sqrt{x}} \, dx \\ &= 2 \cdot \int \frac{1}{2\sqrt{x}} \, dx \\ &= 2\sqrt{x} + c \end{aligned}$$

$$3. \quad \int (\tan x + \cot x)^2 \, dx$$

Solution :

$$\begin{aligned} & \int (\tan x + \cot x)^2 \, dx \\ &= \int (\tan^2 x + 2 \tan x \cdot \cot x + \cot^2 x) \, dx \\ &= \int (\tan^2 x + 2 + \cot^2 x) \, dx \\ &= \int (\sec^2 x - 1 + 2 + \operatorname{cosec}^2 x - 1) \, dx \\ &= \int (\sec^2 x + \operatorname{cosec}^2 x) \, dx \\ &= \int \sec^2 x \, dx + \int \operatorname{cosec}^2 x \, dx \\ &= \tan x + (-\cot x) + c \\ &= \tan x - \cot x + c \end{aligned}$$

$$5. \quad \int \frac{e^{4 \log x} - e^{5 \log x}}{x^5} \, dx$$

Solution :

$$\begin{aligned} & \int \frac{e^{4 \log x} - e^{5 \log x}}{x^5} \, dx \\ &= \int \frac{e^{\log x^4} - e^{\log x^5}}{x^5} \, dx, \quad \because a^{\log_a f(x)} = f(x) \\ &= \int \left(\frac{x^4 - x^5}{x^5} \right) \, dx \\ &= \int \left(\frac{1}{x} - 1 \right) \, dx \\ &= \log(x) - x + c \end{aligned}$$

$$6. \quad \int \frac{2x+3}{5x-1} \, dx$$

Solution : $\frac{N}{D} = Q + \frac{R}{D}$

$$\begin{array}{r} \frac{2}{5} \\ (5x-1) \overline{) 2x+3} \\ \underline{2x - \frac{2}{5}} \\ + \\ \underline{3 + \frac{2}{5}} = \frac{17}{5} \end{array}$$

$$\therefore 2x+3 = \frac{2}{5}(5x-1) + 3 + \frac{2}{5}$$

$$\begin{aligned} I &= \int \left[\frac{2}{5} + \frac{\frac{17}{5}}{5x-1} \right] \, dx \\ &= \frac{2}{5}x + \frac{17}{5} \log(5x-1) \cdot \frac{1}{5} + c \\ &= \frac{2}{5}x + \frac{17}{25} \log(5x-1) + c \end{aligned}$$

7. $\int \frac{1}{\sqrt{3x+1} - \sqrt{3x-5}} dx$

Solution :

$$\begin{aligned} & \int \frac{1}{\sqrt{3x+1} - \sqrt{3x-5}} dx \\ &= \int \left(\frac{1}{\sqrt{3x+1} - \sqrt{3x-5}} \right) \cdot \left(\frac{\sqrt{3x+1} + \sqrt{3x-5}}{\sqrt{3x+1} + \sqrt{3x-5}} \right) dx \\ &= \int \frac{\sqrt{3x+1} + \sqrt{3x-5}}{3x+1 - 3x+5} dx \\ &= \int \frac{\sqrt{3x+1} + \sqrt{3x-5}}{6} dx \\ &= \frac{1}{6} \cdot \int \left((3x+1)^{\frac{1}{2}} + (3x-5)^{\frac{1}{2}} \right) dx \\ &= \frac{1}{6} \cdot \left\{ \int (3x+1)^{\frac{1}{2}} dx + \int (3x-5)^{\frac{1}{2}} dx \right\} \\ &= \frac{1}{6} \cdot \left\{ \frac{(3x+1)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right) \cdot 3} + \frac{(3x-5)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right) \cdot 3} \right\} + c \\ &= \frac{1}{18} \cdot \left\{ \frac{2}{3} (3x+1)^{\frac{3}{2}} + \frac{2}{3} (3x-5)^{\frac{3}{2}} \right\} + c \\ &= \frac{1}{27} \cdot \left\{ (3x+1)^{\frac{3}{2}} + (3x-5)^{\frac{3}{2}} \right\} + c \end{aligned}$$

8. $\int \frac{2x-7}{\sqrt{3x-2}} dx$

Solution : Express $(2x-7)$ in terms of $(3x-2)$

$$\begin{aligned} 2x-7 &= \frac{2}{3}(3x-2) + \frac{4}{3} - 7 \\ &= \frac{2}{3}(3x-2) - \frac{17}{3} \end{aligned}$$

$$\begin{aligned} I &= \int \left[\frac{\frac{2}{3}(3x-2) - \frac{17}{3}}{\sqrt{3x-2}} \right] dx \\ &= \int \left[\frac{\frac{2}{3}(3x-2)}{\sqrt{3x-2}} - \frac{\frac{17}{3}}{\sqrt{3x-2}} \right] dx \\ &= \frac{2}{3} \int \sqrt{3x-2} dx - \frac{17}{3} \int \frac{1}{\sqrt{3x-2}} dx \\ &= \frac{2}{3} \int (3x-2)^{\frac{1}{2}} dx - \frac{17}{3} \int \frac{1}{\sqrt{3x-2}} dx \\ &= \frac{2}{3} \cdot \frac{(3x-2)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} \cdot \frac{1}{3} - \frac{17}{3} \cdot 2 \cdot (\sqrt{3x-2}) \cdot \frac{1}{3} + c \\ &= \frac{4}{27} \cdot (3x-2)^{\frac{3}{2}} - \frac{34}{9} \cdot (3x-2)^{\frac{1}{2}} + c \end{aligned}$$

9. $\int \frac{x^3}{x-1} dx$

Solution :

$$\begin{aligned} I &= \int \frac{x^3-1+1}{x-1} dx \\ &= \int \left(\frac{x^3-1}{x-1} + \frac{1}{x-1} \right) dx \\ &= \int \left(\frac{(x-1)(x^2+x+1)}{(x-1)} + \frac{1}{x-1} \right) dx \\ &= \int \left(x^2+x+1 + \frac{1}{x-1} \right) dx \\ &= \frac{x^3}{3} + \frac{x^2}{2} + x + \log(x-1) + c \end{aligned}$$

10. $\int \frac{3^x - 4^x}{5^x} dx$

Solution :

$$\begin{aligned} I &= \int \left(\frac{3^x}{5^x} - \frac{4^x}{5^x} \right) dx \\ &= \int \left[\left(\frac{3}{5} \right)^x - \left(\frac{4}{5} \right)^x \right] dx \\ &= \frac{\left(\frac{3}{5} \right)^x}{\log \frac{3}{5}} - \frac{\left(\frac{4}{5} \right)^x}{\log \frac{4}{5}} + c \end{aligned}$$

11. $\int \cos^3 x \, dx$

Solution : $\cos 3A = 4 \cos^3 A - 3 \cos A$

$$\begin{aligned} I &= \int \frac{1}{4} (\cos 3x + 3 \cos x) \, dx \\ &= \frac{1}{4} \left(\sin 3x \cdot \frac{1}{3} + 3 \cdot \sin x \right) + c \\ &= \frac{1}{12} \sin 3x + \frac{3}{4} \sin x + c \end{aligned}$$

13. $\int \sin^4 x \, dx$

Solution :

$$\begin{aligned} I &= \int (\sin^2 x)^2 \, dx \\ &= \int \left(\frac{1}{2} (1 - \cos 2x) \right)^2 \, dx \\ &= \frac{1}{4} \cdot \int (1 - 2 \cos 2x + \cos^2 2x) \, dx \\ &= \frac{1}{4} \cdot \int \left[1 - 2 \cos 2x + \frac{1}{2} (1 + \cos 4x) \right] \, dx \\ &= \frac{1}{4} \cdot \int \left(1 - 2 \cos 2x + \frac{1}{2} + \frac{1}{2} \cos 4x \right) \, dx \\ &= \frac{1}{4} \cdot \int \left(\frac{3}{2} - 2 \cos 2x + \frac{1}{2} \cos 4x \right) \, dx \\ &= \frac{1}{4} \cdot \left[\frac{3}{2}x - 2 \sin 2x \cdot \frac{1}{2} + \frac{1}{2} \sin 4x \cdot \frac{1}{4} \right] + c \\ &= \frac{1}{4} \cdot \left[\frac{3}{2}x - \sin 2x + \frac{1}{8} \sin 4x \right] + c \end{aligned}$$

12. $\int \sqrt{1 + \sin 3x} \, dx$

Solution :

$$\begin{aligned} I &= \int \sqrt{\cos^2 \frac{3x}{2} + \sin^2 \frac{3x}{2} + 2 \sin \frac{3x}{2} \cdot \cos \frac{3x}{2}} \, dx \\ &= \int \sqrt{\left(\cos \frac{3x}{2} + \sin \frac{3x}{2} \right)^2} \, dx \\ &= \int \left(\cos \frac{3x}{2} + \sin \frac{3x}{2} \right) \, dx \\ &= \sin \frac{3x}{2} \cdot \frac{1}{\frac{3}{2}} - \cos \frac{3x}{2} \cdot \frac{1}{\frac{3}{2}} + c \\ &= \frac{2}{3} \left(\sin \frac{3x}{2} - \cos \frac{3x}{2} \right) + c \end{aligned}$$

14. $\int \sin 5x \cdot \cos 7x \, dx$

Solution : We know that

$$2 \sin A \cdot \cos B = \sin (A + B) + \sin (A - B)$$

$$\begin{aligned} I &= \frac{1}{2} \int 2 \sin 5x \cdot \cos 7x \, dx \\ &= \frac{1}{2} \int [\sin (5x + 7x) + \sin (5x - 7x)] \, dx \\ &= \frac{1}{2} \int [\sin (12x) + \sin (-2x)] \, dx \\ &= \frac{1}{2} \int (\sin 12x - \sin 2x) \, dx \\ &= \frac{1}{2} \cdot \left[-\cos 12x \cdot \frac{1}{12} + \cos 2x \cdot \frac{1}{2} \right] + c \\ I &= -\frac{1}{24} \cos 12x + \frac{1}{4} \cos 2x + c \end{aligned}$$

15. $\int \frac{\sin^3 x - \cos^3 x}{\sin^2 x \cdot \cos^2 x} \, dx$

Solution : $I = \int \left(\frac{\sin^3 x}{\sin^2 x \cdot \cos^2 x} - \frac{\cos^3 x}{\sin^2 x \cdot \cos^2 x} \right) \, dx$

$$= \int \left(\frac{\sin x}{\cos^2 x} - \frac{\cos x}{\sin^2 x} \right) \, dx$$

$$\begin{aligned} &= \int \left(\frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} - \frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} \right) \, dx \\ &= \int (\sec x \cdot \tan x - \operatorname{cosec} x \cdot \cot x) \, dx \\ &= \sec x - (-\operatorname{cosec} x) + c \\ I &= \sec x + \operatorname{cosec} x + c \end{aligned}$$

16. $\int \frac{1}{1 - \sin x} dx$

Solution :

$$\begin{aligned} I &= \int \left(\frac{1}{1 - \sin x} \right) \left(\frac{1 + \sin x}{1 + \sin x} \right) dx \\ &= \int \frac{1 + \sin x}{1 - \sin^2 x} dx \\ &= \int \frac{1 + \sin x}{\cos^2 x} dx \\ &= \int \left(\frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x} \right) dx \\ &= \int (\sec^2 x + \sec x \cdot \tan x) dx \\ &= \tan x + \sec x + c \end{aligned}$$

17. $\int \left(\frac{\cos x}{1 - \cos x} \right) dx$

Solution :

$$\begin{aligned} I &= \int \left(\frac{\cos x}{1 - \cos x} \right) \left(\frac{1 + \cos x}{1 + \cos x} \right) dx \\ &= \int \frac{\cos x (1 + \cos x)}{1 - \cos^2 x} dx \\ &= \int \left(\frac{\cos x + \cos^2 x}{\sin^2 x} \right) dx \\ &= \int \left(\frac{\cos x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} \right) dx \\ &= \int (\operatorname{cosec} x \cdot \cot x + \cot^2 x) dx \\ &= \int (\operatorname{cosec} x \cdot \cot x + \operatorname{cosec}^2 x - 1) dx \\ &= (-\operatorname{cosec} x) + (-\cot x) - x + c \\ &= -\operatorname{cosec} x - \cot x - x + c \end{aligned}$$

Activity :

18. $\int \frac{\cos x - \cos 2x}{1 - \cos x} dx$

Solution :

$$\begin{aligned} &\int \frac{\cos x - \cos 2x}{1 - \cos x} dx \\ &= \int \frac{\cos x - (\dots\dots\dots)}{1 - \cos x} dx \\ &= \int \frac{\cos x - \dots\dots\dots}{1 - \cos x} dx \\ &= \int \frac{\cos x (1 - \cos x) + \dots\dots\dots}{1 - \cos x} dx \\ &= \int \left[\cos x + \frac{\dots\dots\dots}{1 - \cos x} \right] dx \\ &= \int [\cos x + (1 + \cos x)] dx \\ &= \int (1 + 2 \cos x) dx \\ &= x + 2 \sin x + c \end{aligned}$$

19. $\int \sin^{-1}(\cos 3x) \, dx$

Solution :

$$\begin{aligned} I &= \int \sin^{-1}\left(\sin \frac{\pi}{2} - 3x\right) dx \\ &= \int \left(\frac{\pi}{2} - 3x\right) dx \\ &= \frac{\pi}{2}x - 3\frac{x^2}{2} + c \end{aligned}$$

20. $\int \tan^{-1}\left(\frac{\sin 2x}{1 + \cos 2x}\right) dx$

Solution :

$$\begin{aligned} I &= \int \cot^{-1}\left(\frac{1 + \cos 2x}{\sin 2x}\right) dx \\ &= \int \cot^{-1}\left(\frac{2 \cos^2 x}{2 \sin x \cdot \cos x}\right) dx \\ &= \int \cot^{-1}(\cot x) \, dx \\ &= \int x \, dx = \frac{x^2}{2} + c \end{aligned}$$

21. $\int \tan^{-1} \sqrt{\frac{1 - \sin x}{1 + \sin x}} \, dx$

Solution :

$$\begin{aligned} I &= \int \tan^{-1} \sqrt{\frac{1 - \cos\left(\frac{\pi}{2} - x\right)}{1 + \cos\left(\frac{\pi}{2} - x\right)}} \, dx \\ &= \int \tan^{-1} \sqrt{\frac{2 \sin^2\left(\frac{\pi}{4} - \frac{x}{2}\right)}{2 \cos^2\left(\frac{\pi}{4} - \frac{x}{2}\right)}} \, dx \\ &= \int \tan^{-1} \sqrt{\tan^2\left(\frac{\pi}{4} - \frac{x}{2}\right)} \, dx \\ &= \int \tan^{-1} \left[\tan\left(\frac{\pi}{4} - \frac{x}{2}\right) \right] \, dx \\ &= \int \left(\frac{\pi}{4} - \frac{x}{2}\right) \, dx \\ &= \frac{\pi}{4}x - \frac{1}{2} \cdot \frac{x^2}{2} + c \\ &= \frac{\pi}{4}x - \frac{x^2}{4} + c \end{aligned}$$

EXERCISE 3.1

I. Integrate the following functions w. r. t. x :

$$\begin{aligned} \text{(i)} \quad & x^3 + x^2 - x + 1 & \text{(ii)} \quad & x^2 \left(1 - \frac{2}{x}\right)^2 \\ \text{(iii)} \quad & 3 \sec^2 x - \frac{4}{x} + \frac{1}{x\sqrt{x}} - 7 & & \\ \text{(iv)} \quad & 2x^3 - 5x + \frac{3}{x} + \frac{4}{x^5} & \text{(v)} \quad & \frac{3x^3 - 2x + 5}{x\sqrt{x}} \end{aligned}$$

II. Evaluate :

$$\begin{aligned} \text{(i)} \quad & \int \tan^2 x \, dx & \text{(ii)} \quad & \int \frac{\sin 2x}{\cos x} \, dx \\ \text{(iii)} \quad & \int \frac{\sin x}{\cos^2 x} \, dx & \text{(iv)} \quad & \int \frac{\cos 2x}{\sin^2 x} \, dx \\ \text{(v)} \quad & \int \frac{\cos 2x}{\sin^2 x \cdot \cos^2 x} \, dx & \text{(vi)} \quad & \int \frac{\sin x}{1 + \sin x} \, dx \\ \text{(vii)} \quad & \int \frac{\tan x}{\sec x + \tan x} \, dx & \text{(viii)} \quad & \int \sqrt{1 + \sin 2x} \, dx \\ \text{(ix)} \quad & \int \sqrt{1 - \cos 2x} \, dx & \text{(x)} \quad & \int \sin 4x \cdot \cos 3x \, dx \end{aligned}$$

III. Evaluate :

$$\begin{aligned} \text{(i)} \quad & \int \frac{x}{x+2} \, dx & \text{(ii)} \quad & \int \frac{4x+3}{2x+1} \, dx \\ \text{(iii)} \quad & \int \frac{5x+2}{3x-4} \, dx & \text{(iv)} \quad & \int \frac{x-2}{\sqrt{x+5}} \, dx \\ \text{(v)} \quad & \int \frac{2x-7}{\sqrt{4x-1}} \, dx & \text{(vi)} \quad & \int \frac{\sin 4x}{\cos 2x} \, dx \\ \text{(vii)} \quad & \int \sqrt{1 + \sin 5x} \, dx & \text{(viii)} \quad & \int \cos^2 x \, dx \\ \text{(ix)} \quad & \int \frac{2}{\sqrt{x} - \sqrt{x+3}} \, dx & & \\ \text{(x)} \quad & \int \frac{3}{\sqrt{7x-2} - \sqrt{7x-5}} \, dx & & \end{aligned}$$

IV. $f'(x) = x - \frac{3}{x^3}$, $f(1) = \frac{11}{2}$ then find $f(x)$.

3.2 Methods of integration :

We have evaluated the integrals which can be reduced to standard forms by algebraic or trigonometric simplifications. This year we are going to study three special methods of reducing an integral to a standard form, namely –

1. Integration by substitution
2. Integration by parts
3. Integration by partial fraction

3.2.1 Integration by substitution :

Theorem 1 : If $x = \phi(t)$ is a differentiable function of t , then $\int f(x) dx = \int f[\phi(t)] \phi'(t) dt$.

Proof : $x = \phi(t)$ is a differentiable function of t .

$$\therefore \frac{dx}{dt} = \phi'(t)$$

$$\text{Let } \int f(x) dx = g(x) \Rightarrow \frac{d}{dx} [g(x)] = f(x)$$

By Chain rule,

$$\begin{aligned} \frac{d}{dt} [g(x)] &= \frac{d}{dx} [g(x)] \frac{dx}{dt} \\ &= f(x) \frac{dx}{dt} \\ &= f[\phi(t)] \phi'(t) \end{aligned}$$

By definition of integration,

$$g(x) = \int f[\phi(t)] \phi'(t) dt$$

$$\therefore \int f(x) dx = \int f[\phi(t)] \phi'(t) dt$$

For example 1 : $\int 3x^2 \sin(x^3) dx$

$$\text{Let } x^3 = t$$

$$\therefore 3x^2 dx = dt$$

$$= \int \sin t dt$$

$$= -\cos t + c$$

$$= -\cos(x^3) + c$$

Corollary I :

$$\text{If } \int f(x) \, dx = g(x) + c$$

$$\text{then } \int f(ax+b) \, dx = g(ax+b) \frac{1}{a} + c$$

Proof : Let $I = \int f(ax+b) \, dx$

$$\text{put } ax+b = t$$

Differentiating both the sides

$$a \, dx = 1 \, dt \Rightarrow dx = \frac{1}{a} \, dt$$

$$I = \int f(t) \frac{1}{a} \, dt$$

$$= \frac{1}{a} \cdot \int f(t) \, dt$$

$$= \frac{1}{a} \cdot g(t) + c$$

$$= \frac{1}{a} \cdot g(ax+b) + c$$

$$\therefore \int f(ax+b) \, dx = g(ax+b) \frac{1}{a} + c$$

For example : $\int \sec^2(5x-4) \, dx$

$$= \frac{1}{5} \tan(5x-4) + c$$

Corollary III :

$$\int \frac{f'(x)}{f(x)} \, dx = \log(f(x)) + c$$

Proof : Consider $\int \frac{f'(x)}{f(x)} \, dx$

$$\text{put } f(x) = t$$

Differentiating both the sides

$$f'(x) \, dx = dt$$

$$I = \int \frac{1}{t} \, dt$$

$$= \log(t) + c$$

$$= \log(f(x)) + c$$

$$\therefore \int \frac{f'(x)}{f(x)} \, dx = \log(f(x)) + c$$

Corollary II :

$$\int [f(x)]^n \cdot f'(x) \, dx = \frac{[f(x)]^{n+1}}{n+1} + c, n \neq -1$$

Proof : Let $I = \int [f(x)]^{n+1} \cdot f'(x) \, dx$

$$\text{put } f(x) = t$$

Differentiating both the sides

$$f'(x) \, dx = dt$$

$$I = \int [t]^n \, dt$$

$$= \frac{t^{n+1}}{n+1} + c, \quad n \neq -1$$

$$= \frac{[f(x)]^{n+1}}{n+1} + c$$

$$\therefore \int [f(x)]^n \cdot f'(x) \, dx = \frac{[f(x)]^{n+1}}{n+1} + c$$

For example : $\int \frac{(\sin^{-1} x)^3}{\sqrt{1-x^2}} \, dx$

$$= \int [(\sin^{-1} x)^3] \left(\frac{1}{\sqrt{1-x^2}} \right) dx$$

$$= \frac{(\sin^{-1} x)^4}{4} + c$$

For example : $\int \cot x \, dx$

$$= \int \frac{\cos x}{\sin x} \, dx$$

$$\frac{d}{dx} \sin x = \cos x$$

$$= \int \left(\frac{\frac{d}{dx} \sin x}{\sin x} \, dx \right)$$

$$= \log(\sin x) + c$$

Corollary IV :

$$\int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + c$$

Proof : Consider $\int \frac{f'(x)}{\sqrt{f(x)}} dx$

put $f(x) = t$

Differentiating both the sides

$$f'(x) dx = dt$$

$$\begin{aligned} \text{I} &= \int \frac{1}{\sqrt{t}} dt \\ &= 2 \cdot \int \frac{1}{2\sqrt{t}} dt \\ &= 2\sqrt{t} + c \\ &= 2\sqrt{f(x)} + c \end{aligned}$$

$$\therefore \int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + c$$

Using corollary III, $\int \frac{f'(x)}{f(x)} dx = \log(f(x)) + c$ we find the integrals of some trigonometric functions.

For example :

$$\begin{aligned} &\int \frac{1}{x\sqrt{\log x}} dx \\ &= \int \left(\frac{\frac{1}{x}}{\sqrt{\log x}} \right) dx \\ &= \int \left(\frac{\frac{d}{dx} \log x}{\sqrt{\log x}} \right) dx \\ &= 2\sqrt{\log x} + c \end{aligned}$$

3.2.2 Integrals of trigonometric functions :

1. $\int \tan x \, dx$

Solution :

$$\begin{aligned} \text{I} &= \int \tan x \, dx \\ &= \int \frac{\sin x}{\cos x} \, dx \\ &= - \int \frac{-\sin x}{\cos x} \, dx \\ &= -\log(\cos x) + c \\ &= \log(\sec x) + c \end{aligned}$$

Activity :

2. $\int \cot(5x - 4) \, dx$

Solution :

$$\begin{aligned} \text{I} &= \int \frac{\dots\dots\dots}{\sin(5x - 4)} \, dx \\ &= \frac{1}{\dots\dots} \int \frac{5 \cos(5x - 4)}{\dots\dots\dots} \, dx \\ \frac{d}{dx} (\dots\dots\dots) &= \dots\dots\dots \end{aligned}$$

$$= \frac{1}{5} \log[\sin(5x - 4)] + c$$

3. $\int \sec x \, dx = \log (\sec x + \tan x) + c$

Solution : Let $I = \int \sec x \, dx$

$$= \int \frac{(\sec x)(\sec x + \tan x)}{\sec x + \tan x} \, dx$$

$$= \int \frac{\sec^2 x + \sec x \cdot \tan x}{\sec x + \tan x} \, dx$$

$$= \int \frac{\sec x \cdot \tan x + \sec^2 x}{\sec x + \tan x} \, dx$$

$$\therefore \frac{d}{dx} (\sec x + \tan x) = \sec x \cdot \tan x + \sec^2 x$$

$$\therefore \int \sec x \, dx = \log (\sec x + \tan x) + c$$

Also,

$$\int \sec x \, dx = \log \left[\tan \left(\frac{x}{2} + \frac{\pi}{4} \right) \right] + c$$

Activity :

4. $\int \operatorname{cosec} x \, dx = \log (\operatorname{cosec} x - \cot x) + c$

Solution : Let $I = \int \operatorname{cosec} x \, dx$

$$= \int \frac{(\operatorname{cosec} x)(\dots\dots\dots)}{(\dots\dots\dots)} \, dx$$

$$= \int \frac{\dots\dots\dots}{\dots\dots\dots} \, dx$$

$$= \int \frac{-\operatorname{cosec} x \cdot \cot x + \operatorname{cosec}^2 x}{\dots\dots\dots} \, dx$$

$$\frac{d}{dx} (\operatorname{cosec} x - \cot x)$$

$$= \dots\dots\dots$$

$$= \log (\operatorname{cosec} x - \cot x) + c$$

$$\therefore \int \operatorname{cosec} x \, dx = \log (\operatorname{cosec} x - \cot x) + c$$

Also,

$$\int \operatorname{cosec} x \, dx = \log \left(\tan \frac{x}{2} \right) + c$$

SOLVED EXAMPLES

Ex. : Evaluate the following functions :

1. $\int \frac{\cot (\log x)}{x} \, dx$

Solution : Let $I = \int \frac{\cot (\log x)}{x} \, dx$

$$\text{put } \log x = t$$

$$\therefore \frac{1}{x} \, dx = 1 \, dt$$

$$= \int \cot t \, dt$$

$$= \log (\sin t) + c$$

$$= \log (\sin \log x) + c$$

2. $\int \frac{\cos \sqrt{x}}{\sqrt{x}} \, dx$

Solution : Let $I = \int \frac{\cos \sqrt{x}}{\sqrt{x}} \, dx$

$$\text{put } \sqrt{x} = t$$

$$\therefore \frac{1}{2\sqrt{x}} \, dx = 1 \, dt$$

$$\therefore \frac{1}{\sqrt{x}} \, dx = 2 \, dt$$

$$= 2 \cdot \int \cos t \, dt$$

$$= 2 \cdot \sin t + c$$

$$= 2 \cdot \sin \sqrt{x} + c$$

3. $\int \frac{\sec^8 x}{\operatorname{cosec} x} dx$

Solution : $I = \int \sec^7 x \cdot \sec x \cdot \frac{1}{\operatorname{cosec} x} dx$
 $= \int \sec^7 x \cdot \frac{1}{\cos x} \cdot \sin x dx$
 $= \int \sec^7 x \cdot \tan x dx$
 $= \int \sec^6 x \cdot \sec x \cdot \tan x dx$
 put $\sec x = t$
 $\therefore \sec x \cdot \tan x dx = dt$
 $= \int t^6 dt$
 $= \frac{t^7}{7} + c$
 $= \frac{\sec^7 x}{7} + c$

5. $\int 5^{5^x} \cdot 5^x dx$

Solution : $I = \int 5^{5^x} \cdot 5^x dx$
 put $5^x = t$
 $\therefore 5^x \cdot \log 5 dx = 1 dt$
 $5^x dx = \frac{1}{\log 5} dt$
 $= \int 5^t \cdot \frac{1}{\log 5} dt$
 $I = \frac{1}{\log 5} \cdot \int 5^t dt$
 $= \frac{1}{\log 5} \cdot 5^t \cdot \frac{1}{\log 5} + c$
 $= \left(\frac{1}{\log 5} \right)^2 \cdot 5^{5^x} + c$

7. $\int \frac{e^x (1+x)}{\cos (x \cdot e^x)} dx$

Solution : put $x \cdot e^x = t$
 Differentiating both sides
 $(x \cdot e^x + e^x \cdot 1) dx = 1 dt$
 $e^x (1+x) dx = 1 dt$

4. $\int \frac{1}{x + \sqrt{x}} dx$

Solution : $I = \int \frac{1}{x + \sqrt{x}} dx$
 $= \int \frac{1}{\sqrt{x}(\sqrt{x} + 1)} dx$
 put $\sqrt{x} + 1 = t$
 $\therefore \frac{1}{2\sqrt{x}} dx = 1 dt$
 $\therefore \frac{1}{\sqrt{x}} dx = 2 dt$
 $= \int \frac{1}{t} \cdot 2 dt$
 $= 2 \cdot \int \frac{1}{t} dt$
 $= 2 \cdot \log(t) + c$
 $= 2 \cdot \log(\sqrt{x} + 1) + c$

6. $\int \frac{1}{1 + e^{-x}} dx$

Solution : $I = \int \frac{1}{1 + e^{-x}} dx$
 $= \int \frac{1}{1 + \frac{1}{e^x}} dx$
 $= \int \frac{1}{\frac{e^x + 1}{e^x}} dx$
 $= \int \frac{e^x}{e^x + 1} dx$
 $\therefore \frac{d}{dx} (e^x + 1) = e^x$
 $= \log [e^x + 1] + c$

$I = \int \frac{1}{\cos t} dt$
 $= \int \sec t dt$
 $= \log (\sec t + \tan t) + c$
 $= \log (\sec (xe^x) + \tan (xe^x)) + c$

8. $\int \frac{1}{3x + 7x^{-n}} dx$

Solution : Consider $\int \frac{1}{3x + 7x^{-n}} dx$

$$= \int \frac{1}{3x + \frac{7}{x^n}} dx = \int \frac{1}{\frac{3x^{n+1} + 7}{x^n}} dx$$

$$= \int \frac{x^n}{3x^{n+1} + 7} dx$$

put $3x^{n+1} + 7 = t$
 Differentiate w. r. t. x
 $3(n+1)x^n dx = dt$
 $\therefore x^n dx = \frac{1}{3(n+1)} dt$

$$= \int \frac{1}{\frac{3(n+1)}{t}} dt$$

$$= \frac{1}{3(n+1)} \cdot \log(t) + c$$

$$= \frac{1}{3(n+1)} \cdot \log(3x^{n+1} + 7) + c$$

9. $\int (3x + 2) \sqrt{x-4} dx$

Solution : put $x - 4 = t$
 $\therefore x = 4 + t$
 Differentiate
 $1 dx = 1 dt$

$$= \int [3(4 + t) + 2] \cdot \sqrt{t} dt$$

$$= \int (14 + 3t) t^{\frac{1}{2}} dt$$

$$= \int \left(14t^{\frac{1}{2}} + 3t^{\frac{3}{2}} \right) dt$$

$$= 14 \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + 3 \frac{t^{\frac{5}{2}}}{\frac{5}{2}}$$

$$= \frac{28}{3} (x-4)^{\frac{3}{2}} + \frac{6}{5} (x-4)^{\frac{5}{2}} + c$$

10. $\int \frac{\sin(x+a)}{\cos(x-b)} dx$

Solution :

$$= \int \frac{\sin[(x-b) + (a+b)]}{\cos(x-b)} dx$$

$$= \int \frac{\sin(x-b) \cdot \cos(a+b) + \cos(x-b) \cdot \sin(a+b)}{\cos(x-b)} dx$$

$$= \int \left[\frac{\sin(x-b) \cdot \cos(a+b)}{\cos(x-b)} + \frac{\cos(x-b) \cdot \sin(a+b)}{\cos(x-b)} \right] dx$$

$$= \int [\cos(a+b) \cdot \tan(x-b) + \sin(a+b)] dx$$

$$= \cos(a+b) \cdot \log(\sec(x-b)) + x \cdot \sin(a+b) + c$$

11. $\int \frac{e^x + 1}{e^x - 1} dx$

Solution :

$$\begin{aligned} I &= \int \frac{e^x - 1 + 2}{e^x - 1} dx \\ &= \int \left(\frac{e^x - 1}{e^x - 1} + \frac{2}{e^x - 1} \right) dx \\ &= \int \left(1 + \frac{2}{e^x - 1} \right) dx \\ &= \int dx + \int \frac{2}{e^x(1 - e^{-x})} dx \\ &= \int 1 dx + 2 \int \frac{e^{-x}}{1 - e^{-x}} dx \end{aligned}$$

put $(1 - e^{-x}) = t$

Differentiate w. r. t. x

$$-(e^{-x})(-1) dx = 1 dt$$

$$e^{-x} dx = 1 dt$$

$$I = \int 1 dx + 2 \int \frac{1}{t} dt$$

$$= x + 2 \cdot \log(t) + c$$

$$= x + 2 \log(1 - e^{-x}) + c$$

$$\therefore \int \frac{e^x + 1}{e^x - 1} dx = x + 2 \log(1 - e^{-x}) + c$$

12. $\int \frac{1}{1 - \tan x} dx$

Solution :

$$\begin{aligned} I &= \int \frac{1}{1 - \frac{\sin x}{\cos x}} dx \\ &= \int \frac{\cos x}{\cos x - \sin x} dx \\ &= \int \frac{\cos x}{\sqrt{2} \left(\frac{1}{\sqrt{2}} \cos x - \frac{1}{\sqrt{2}} \sin x \right)} dx \end{aligned}$$

$$= \frac{1}{\sqrt{2}} \int \frac{\cos x}{\cos \frac{\pi}{4} \cos x - \sin \frac{\pi}{4} \sin x} dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{\cos x}{\cos \left(x + \frac{\pi}{4} \right)} dx$$

put $x + \frac{\pi}{4} = t \therefore x = t - \frac{\pi}{4}$

Differentiating both sides

$$1 dx = 1 dt$$

$$= \frac{1}{\sqrt{2}} \int \frac{\cos \left(t - \frac{\pi}{4} \right)}{\cos t} dt$$

$$= \frac{1}{\sqrt{2}} \int \frac{\cos t \cdot \cos \frac{\pi}{4} + \sin t \cdot \sin \frac{\pi}{4}}{\cos t} dt$$

$$= \frac{1}{\sqrt{2}} \int \left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \tan t \right] dt$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} [t + \log(\sec t)] + c$$

$$= \frac{1}{2} \left[x + \frac{\pi}{4} + \log \sec \left(x + \frac{\pi}{4} \right) \right] + c$$

To evaluate the integrals of type $\int \frac{a \cos x + b \sin x}{c \cos x + d \sin x} dx$, express the Numerator as $N = \lambda (D) + \mu (D)'$, find the constants λ & μ by comparing the coefficients of like terms and then integrate the function.

EXERCISE 3.2 (A)

I. Integrate the following functions w. r. t. x :

1. $\frac{(\log x)^n}{x}$
2. $\frac{(\sin^{-1} x)^{\frac{3}{2}}}{\sqrt{1-x^2}}$
3. $\frac{1+x}{x \cdot \sin(x + \log x)}$
4. $\frac{x \cdot \sec^2(x^2)}{\sqrt{\tan^3(x^2)}}$
5. $\frac{e^{3x}}{e^{3x} + 1}$
6. $\frac{(x^2 + 2)}{(x^2 + 1)} \cdot a^{x + \tan^{-1} x}$
7. $\frac{e^x \cdot \log(\sin e^x)}{\tan(e^x)}$
8. $\frac{e^{2x} + 1}{e^{2x} - 1}$
9. $\sin^4 x \cdot \cos^3 x$
10. $\frac{1}{4x + 5x^{-11}}$
11. $x^9 \cdot \sec^2(x^{10})$
12. $e^{3 \log x} \cdot (x^4 + 1)^{-1}$
13. $\frac{\sqrt{\tan x}}{\sin x \cdot \cos x}$
14. $\frac{(x-1)^2}{(x^2 + 1)^2}$
15. $\frac{2 \sin x \cdot \cos x}{3 \cos^2 x + 4 \sin^2 x}$
16. $\frac{1}{\sqrt{x} + \sqrt{x^3}}$
17. $\frac{10x^9 + 10^x \cdot \log 10}{10^x + x^{10}}$
18. $\frac{x^{n-1}}{\sqrt{1+4x^n}}$
19. $(2x+1)\sqrt{x+2}$
20. $x^5 \cdot \sqrt{a^2 + x^2}$
21. $(5-3x)(2-3x)^{-\frac{1}{2}}$
22. $\frac{7+4x+5x^2}{(2x+3)^{\frac{3}{2}}}$

23. $\frac{x^2}{\sqrt{9-x^6}}$
24. $\frac{1}{x(x^3-1)}$
25. $\frac{1}{x \cdot \log x \cdot \log(\log x)}$

II. Integrate the following functions w. r. t. x :

1. $\frac{\cos 3x - \cos 4x}{\sin 3x + \sin 4x}$
2. $\frac{\cos x}{\sin(x-a)}$
3. $\frac{\sin(x-a)}{\cos(x+b)}$
4. $\frac{1}{\sin x \cdot \cos x + 2 \cos^2 x}$
5. $\frac{\sin x + 2 \cos x}{3 \sin x + 4 \cos x}$
6. $\frac{1}{2 + 3 \tan x}$
7. $\frac{4e^x - 25}{2e^x - 5}$
8. $\frac{20 + 12e^x}{3e^x + 4}$
9. $\frac{3e^{2x} + 5}{4e^{2x} - 5}$
10. $\cos^8 x \cdot \cot x$
11. $\tan^5 x$
12. $\cos^7 x$
13. $\tan 3x \cdot \tan 2x \cdot \tan x$
14. $\sin^5 x \cdot \cos^8 x$
15. $3^{\cos^2 x} \cdot \sin 2x$
16. $\frac{\sin 6x}{\sin 10x \cdot \sin 4x}$
17. $\frac{\sin x \cdot \cos^3 x}{1 + \cos^2 x}$

3.2.3 Some Special Integrals

1. $\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$
2. $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left(\frac{x-a}{x+a} \right) + c$
3. $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \left(\frac{a+x}{a-x} \right) + c$
4. $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \left(\frac{x}{a} \right) + c$
5. $\int \frac{1}{\sqrt{x^2 - a^2}} dx = \log(x + \sqrt{x^2 - a^2}) + c$
6. $\int \frac{1}{\sqrt{x^2 + a^2}} dx = \log(x + \sqrt{x^2 + a^2}) + c$
7. $\int \frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a} \sec^{-1} \left(\frac{x}{a} \right) + c$

While evaluating an integral there is no unique substitution, we can use some standard substitutions and try.

No.	Function	Substitution
1.	$\sqrt{a^2 - x^2}$	$x = a \sin \theta$ ($x = a \cos \theta$ can also be used.)
2.	$\sqrt{a^2 + x^2}$	$x = a \tan \theta$
3.	$\sqrt{x^2 - a^2}$	$x = a \sec \theta$
4.	$\sqrt{\frac{a-x}{a+x}}$	$x = a \cos 2\theta$

1. $\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$

Proof :

Let $I = \int \frac{1}{x^2 + a^2} dx$

put $x = a \cdot \tan \theta \Rightarrow \tan \theta = \frac{x}{a}$

i.e. $\theta = \tan^{-1} \left(\frac{x}{a} \right)$

$\therefore dx = a \cdot \sec^2 \theta \cdot d\theta$

$I = \int \frac{1}{a^2 \cdot \tan^2 \theta + a^2} \cdot a \cdot \sec^2 \theta \cdot d\theta$

$= \int \frac{a \cdot \sec^2 \theta}{a^2 (\tan^2 \theta + 1)} d\theta$

$= \int \frac{\sec^2 \theta}{a \cdot \sec^2 \theta} d\theta$

$= \frac{1}{a} \int d\theta$

$= \frac{1}{a} \theta + c$

$= \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$

$\therefore \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$

e.g. $\int \frac{1}{x^2 + 5^2} dx = \frac{1}{5} \tan^{-1} \left(\frac{x}{5} \right) + c$

Alternatively

Consider,

$$\frac{d}{dx} \left[\frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c \right]$$

$$= \frac{d}{dx} \left[\frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) \right] + \frac{d}{dx} c$$

$$= \frac{1}{a} \cdot \frac{1}{1 + \left(\frac{x}{a} \right)^2} \cdot \frac{d}{dx} \left(\frac{x}{a} \right) + 0$$

$$= \frac{1}{a} \cdot \frac{1}{1 + \frac{x^2}{a^2}} \cdot \frac{1}{a}$$

$$= \frac{1}{a^2} \cdot \frac{1}{\frac{a^2 + x^2}{a^2}}$$

$$= \frac{1}{x^2 + a^2}$$

Therefore,

by definition of integration

$$\therefore \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$$

$$2. \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left(\frac{x-a}{x+a} \right) + c$$

Proof :

$$\begin{aligned} \text{Let } I &= \int \frac{1}{x^2 - a^2} dx \\ &= \int \frac{1}{(x+a)(x-a)} dx \\ &= \int \frac{1}{2a} \left[\frac{1}{x-a} - \frac{1}{x+a} \right] dx \\ &= \frac{1}{2a} \int \left[\frac{1}{x-a} - \frac{1}{x+a} \right] dx \\ &= \frac{1}{2a} [\log(x-a) - \log(x+a)] + c \\ &= \frac{1}{2a} \log \left(\frac{x-a}{x+a} \right) + c \end{aligned}$$

$$\therefore \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left(\frac{x-a}{x+a} \right) + c$$

e.g. $\int \frac{1}{x^2 - 9} dx = \frac{1}{2(3)} \log \left(\frac{x-3}{x+3} \right) + c$

Activity :

$$3. \int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \left(\frac{a+x}{a-x} \right) + c$$

Proof : Consider,

$$\begin{aligned} I &= \int \frac{1}{a^2 - x^2} dx \\ &= \int \frac{1}{(\dots)(\dots)} dx \\ &= \int \frac{1}{2a} \left[\frac{1}{\dots} + \frac{1}{\dots} \right] dx \\ &= \frac{1}{2a} \int \left[\frac{\dots}{\dots} + \frac{1}{a+x} \right] dx \\ &= \frac{1}{2a} [\log(a+x) - \log(a-x)] + c \\ &= \frac{1}{2a} \log \left(\frac{\dots}{\dots} \right) + c \end{aligned}$$

$$\therefore \int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \left(\frac{a+x}{a-x} \right) + c$$

e.g. $\int \frac{1}{16 - x^2} dx = \frac{1}{2(4)} \log \left(\frac{4+x}{4-x} \right) + c$

$$4. \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \left(\frac{x}{a} \right) + c$$

Proof :

$$\begin{aligned} \text{Let } I &= \int \frac{1}{\sqrt{a^2 - x^2}} dx \\ \text{put } x &= a \sin \theta \Rightarrow \sin \theta = \frac{x}{a} \\ \therefore \theta &= \sin^{-1} \left(\frac{x}{a} \right) \\ dx &= a \cos \theta d\theta \\ I &= \int \frac{1}{\sqrt{a^2 - a^2 \sin^2 \theta}} a \cos \theta d\theta \\ I &= \int \frac{a \cdot \cos \theta}{a \sqrt{1 - \sin^2 \theta}} d\theta \end{aligned}$$

$$\begin{aligned} &= \int \frac{\cos \theta}{\cos \theta} d\theta \\ &= \int 1 d\theta \\ &= \theta + c \end{aligned}$$

$$\therefore \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \left(\frac{x}{a} \right) + c$$

e.g. $\int \frac{1}{\sqrt{81 - x^2}} dx = \sin^{-1} \left(\frac{x}{9} \right) + c$

Proof : Let $I = \int \frac{1}{\sqrt{x^2 - a^2}} dx$

$$\therefore dx = a \sec \theta \tan \theta d\theta$$

$$= \int \frac{a \cdot \sec \theta \cdot \tan \theta}{\sqrt{a^2 (\sec^2 \theta - 1)}} d\theta$$

$$= \int \frac{a \cdot \sec \theta \cdot \tan \theta}{\sqrt{a^2 \cdot \tan^2 \theta}} d\theta$$

$$= \int \frac{a \cdot \sec \theta \cdot \tan \theta}{a \cdot \tan \theta} d\theta$$

$$= \int \sec \theta \, d\theta$$

$$= \log (\sec \theta + \tan \theta) + c$$

$$= \log (\sec \theta + \sqrt{\sec^2 \theta - 1}) + c$$

$$= \log \left(\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1} \right) + c_1$$

$$= \log \left(\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1} \right) + c_1$$

$$= \log \left(\frac{x + \sqrt{x^2 - a^2}}{a} \right) + c_1$$

$$= \log(x + \sqrt{x^2 - a^2}) - \log a + c_1$$

$$= \log \left(x + \sqrt{x^2 - a^2} \right) + c$$

where $c = c_1 - \log a$

$$\therefore \int \frac{1}{\sqrt{x^2 - a^2}} dx = \log(x + \sqrt{x^2 - a^2}) + c$$

e.g. $\int \frac{1}{\sqrt{x^2 - 16}} dx = \log(x + \sqrt{x^2 - 16}) + c$

6. $\int \frac{1}{\sqrt{a^2 + x^2}} dx = \log(x + \sqrt{a^2 + x^2}) + c$

Proof : use substitution $x = a \tan \theta$

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7. $\int \frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a} \sec^{-1}\left(\frac{x}{a}\right) + c$

Proof : Let $I = \int \frac{1}{x\sqrt{x^2 - a^2}} dx$

$$\text{put } x = a \sec \theta \Rightarrow \theta = \sec^{-1} \left(\frac{x}{a} \right)$$

$$\therefore dx = a \sec \theta \tan \theta d\theta$$

$$\text{I} = \int \frac{a \sec \theta \tan \theta \, dx}{a \sec \theta \sqrt{\dots - a^2}} \dots\dots\dots$$

$$= \int \frac{\tan \theta}{\sqrt{a^2 (\dots)}} d\theta$$

$$= \frac{1}{a} \int 1 \, d\theta$$

$$= \frac{1}{a} \theta + c$$

$$= \frac{1}{a} \sec^{-1} \left(\frac{x}{a} \right) + c$$

$$\therefore \int \frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a} \sec^{-1}\left(\frac{x}{a}\right) + c$$

e.g. $\int \frac{1}{x\sqrt{x^2-64}} dx = \frac{1}{8} \sec^{-1}\left(\frac{x}{8}\right) + c$

3.2.4

In order to evaluate the integrals of type $\int \frac{1}{ax^2 + bx + c} dx$ and $\int \frac{1}{\sqrt{ax^2 + bx + c}} dx$ we can use the following steps.

- (1) Write $ax^2 + bx + c$ as, $a \left(x^2 + \frac{b}{a}x + \frac{c}{a} \right)$, $a > 0$ and take a or $\sqrt{a}$ out of the integral sign.
- (2) $\left(x^2 + \frac{b}{a}x \right)$ or $\left(\frac{b}{a}x - x^2 \right)$ is expressed by the method of completing square by adding and subtracting $\left(\frac{1}{2} \text{ coefficient of } x \right)^2$.
- (3) Express the quadratic expression as a sum or difference of two squares
i.e. $((x + \beta)^2 \pm \alpha^2)$ or $(\alpha^2 - (x + \beta)^2)$
- (4) We know that $\int f(x) dx = g(x) + c \Rightarrow \int f(x + \beta) dx = g(x + \beta) + c$
 $\int f(\alpha x + \beta) dx = \frac{1}{\alpha} g(\alpha x + \beta) + c$
- (5) Use the standard integral formula and express the result in terms of x .

3.2.5

In order to evaluate the integral of type $\int \frac{1}{a \sin^2 x + b \cos^2 x + c} dx$ we can use the following steps.

- (1) Divide the numerator and denominator by $\cos^2 x$ or $\sin^2 x$.
- (2) In denominator replace $\sec^2 x$ by $1 + \tan^2 x$ and /or $\operatorname{cosec}^2 x$ by $1 + \cot^2 x$, if exists.
- (3) Put $\tan x = t$ or $\cot x = t$ so that the integral reduces to the form $\int \frac{1}{at^2 + bt + c} dt$
- (4) Use the standard integral formula and express the result in terms of x .

3.2.6

To evaluate the integral of the form $\int \frac{1}{a \sin x + b \cos x + c} dx$, we use the standard substitution

$$\tan \frac{x}{2} = t.$$

$$\text{If } \tan \frac{x}{2} = t \quad \text{then} \quad \text{(i)} \quad \sec^2 \frac{x}{2} \frac{1}{2} dx = 1 dt$$

$$\text{i.e. } dx = \frac{2}{\sec^2 \frac{x}{2}} dt = \frac{2}{1 + \tan^2 \frac{x}{2}} dt = \frac{2 dt}{1 + t^2}$$

$$\text{(ii)} \quad \sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} = \frac{2t}{1 + t^2}$$

$$(iii) \cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} = \frac{1 - t^2}{1 + t^2}$$

We put $\tan x = t$ for the integral of the type $\int \frac{1}{a \sin 2x + b \cos 2x + c} dx$

therefore $dx = \frac{1}{1 + t^2} dt$

$$\sin 2x = \frac{2t}{1 + t^2}$$

and $\cos 2x = \frac{1 - t^2}{1 + t^2}$

With this substitution the integral reduces to the form $\int \frac{1}{ax^2 + bx + c} dx$. Now use the standard integral formula and express the result in terms of x .



SOLVED EXAMPLES

Ex. : Evaluate :

1. $\int \frac{1}{4x^2 + 11} dx$

Solution : $I = \int \frac{1}{4\left(x^2 + \frac{11}{4}\right)} dx$

$$= \frac{1}{4} \int \frac{1}{x^2 + \left(\frac{\sqrt{11}}{2}\right)^2} dx$$

$$\therefore \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$$

$$I = \frac{1}{4} \left[\frac{1}{\left(\frac{\sqrt{11}}{2}\right)} \tan^{-1} \left[\frac{x}{\left(\frac{\sqrt{11}}{2}\right)} \right] \right] + c$$

$$= \frac{1}{2\sqrt{11}} \tan^{-1} \left(\frac{2x}{\sqrt{11}} \right) + c$$

2. $\int \frac{1}{a^2 - b^2 x^2} dx$

Solution : $I = \int \frac{1}{b^2 \left(\frac{a^2}{b^2} - x^2 \right)} dx$

$$= \frac{1}{b^2} \int \frac{1}{\left(\frac{a}{b} \right)^2 - x^2} dx$$

$$\therefore \int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \left(\frac{a+x}{a-x} \right) + c$$

$$I = \frac{1}{b^2} \frac{1}{2 \left(\frac{a}{b} \right)} \log \left(\frac{\frac{a}{b} + x}{\frac{a}{b} - x} \right) + c$$

$$= \frac{1}{b^2} \frac{1}{2 \left(\frac{a}{b} \right)} \log \left(\frac{\frac{a}{b} + x}{\frac{a}{b} - x} \right) + c$$

$$= \frac{1}{2ab} \log \left(\frac{a + bx}{a - bx} \right) + c$$

3. $\int \frac{1}{\sqrt{3x^2-7}} dx$

Solution : $I = \int \frac{1}{\sqrt{3\left(x^2 - \frac{7}{3}\right)}} dx$

$$= \int \frac{1}{\sqrt{3} \sqrt{x^2 - \left(\frac{\sqrt{7}}{\sqrt{3}}\right)^2}} dx$$

$$= \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{x^2 - \left(\frac{\sqrt{7}}{\sqrt{3}}\right)^2}} dx$$

$$\therefore \int \frac{1}{\sqrt{x^2 - a^2}} dx = \log \left| x + \sqrt{x^2 - a^2} \right| + c$$

$$I = \frac{1}{\sqrt{3}} \log \left(x + \sqrt{x^2 - \left(\frac{\sqrt{7}}{\sqrt{3}}\right)^2} \right) + c$$

$$= \frac{1}{\sqrt{3}} \log \left(x + \sqrt{x^2 - \frac{7}{3}} \right) + c$$

5. $\int \frac{1}{\sqrt{3x^2-4x+2}} dx$

Solution : $= \int \frac{1}{\sqrt{3\left(x^2 - \frac{4}{3}x + \frac{2}{3}\right)}} dx$

$$\therefore \left\{ \left(\frac{1}{2} \text{ coefficient of } x \right)^2 = \left(\frac{1}{2} \left(-\frac{4}{3} \right) \right)^2 = \left(-\frac{2}{3} \right)^2 = \frac{4}{9} \right\}$$

$$= \int \frac{1}{\sqrt{3} \sqrt{x^2 - \frac{4}{3}x + \frac{4}{9} - \frac{4}{9} + \frac{2}{3}}} dx$$

$$= \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{\left(x^2 - \frac{4}{3}x + \frac{4}{9}\right) + \left(\frac{2}{3} - \frac{4}{9}\right)}} dx$$

$$= \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{\left(x - \frac{2}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^2}} dx$$

4. $\int \frac{1}{x^2+8x+12} dx$

Solution : $I = \int \frac{1}{x^2+8x+16-4} dx$

$$= \int \frac{1}{(x+4)^2 - (2)^2} dx$$

$$\therefore \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left(\frac{x-a}{x+a} \right) + c$$

$$I = \frac{1}{2(2)} \log \left(\frac{(x+4)-2}{(x+4)+2} \right) + c$$

$$= \frac{1}{4} \log \left(\frac{x+2}{x+6} \right) + c$$

$$\therefore \int \frac{1}{x^2+8x+12} dx = \frac{1}{4} \log \left(\frac{x+2}{x+6} \right) + c$$

6. $\int \frac{1}{3 - 10x - 25x^2} dx$

Solution :

$$\begin{aligned}
 I &= \int \frac{1}{25 \left(\frac{3}{25} - \frac{10}{25}x - x^2 \right)} dx \\
 &= \int \frac{1}{25 \left[\frac{3}{25} - \left(x^2 + \frac{2}{5}x \right) \right]} dx \\
 \because \left\{ \left(\frac{1}{2} \text{ coefficient of } x \right)^2 \right. \\
 &= \left. \left(\frac{1}{2} \left(\frac{2}{5} \right) \right)^2 = \left(\frac{1}{5} \right)^2 = \frac{1}{25} \right\} \\
 &= \frac{1}{25} \cdot \int \frac{1}{\frac{3}{25} - \left(x^2 - \frac{2}{5}x + \frac{1}{25} - \frac{1}{25} \right)} dx \\
 &= \frac{1}{25} \cdot \int \frac{1}{\frac{3}{25} - \left(x^2 - \frac{2}{5}x + \frac{1}{25} \right) + \frac{1}{25}} dx \\
 &= \frac{1}{25} \cdot \int \frac{1}{\frac{4}{25} - \left(x^2 - \frac{2}{5}x + \frac{1}{25} \right)} dx \\
 &= \frac{1}{25} \cdot \int \frac{1}{\left(\frac{2}{5} \right)^2 - \left(x - \frac{1}{5} \right)^2} dx \\
 \therefore \int \frac{1}{a^2 - x^2} dx &= \frac{1}{2a} \log \left(\frac{a+x}{a-x} \right) + c \\
 I &= \frac{1}{25} \cdot \frac{1}{2 \left(\frac{2}{5} \right)} \cdot \log \left(\frac{\frac{2}{5} + \left(x - \frac{1}{5} \right)}{\frac{2}{5} - \left(x - \frac{1}{5} \right)} \right) + c \\
 &= \frac{1}{5} \cdot \log \left(\frac{1+5x}{3-5x} \right) + c
 \end{aligned}$$

Activity :

7. $\int \frac{1}{\sqrt{1+x-x^2}} dx$

Solution : $I = \int \frac{1}{\sqrt{1 - \left(\dots \dots \dots \right)^2}} dx$

$$\begin{aligned}
 \because \left\{ \left(\frac{1}{2} \text{ coefficient of } x \right)^2 \right. \\
 &= \left. \left(\frac{1}{2} (-1) \right)^2 = \left(-\frac{1}{2} \right)^2 = \frac{1}{4} \right\} \\
 &= \int \frac{1}{\sqrt{1 - \left(x^2 - x + \frac{1}{4} - \frac{1}{4} \right)}} dx \\
 &= \int \frac{1}{\sqrt{1 - \left(\dots \dots \dots \right)^2}} dx
 \end{aligned}$$

$$I = \int \frac{1}{\sqrt{\left(\frac{\sqrt{5}}{2} \right)^2 - \left(x - \frac{1}{2} \right)^2}} dx$$

$$\therefore \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \left(\frac{x}{a} \right) + c$$

$$I = \dots \dots \dots$$

$$= \dots \dots \dots$$

8. $\int \frac{\sin 2x}{3 \sin^4 x - 4 \sin^2 x + 1} dx$

Solution : $I = \int \frac{\sin 2x}{3 (\sin^2 x)^2 - 4 (\sin^2 x) + 1} dx$

put $\sin^2 x = t$ $\therefore 2 \sin x \cos x dx = 1 dt$
 $\therefore \sin 2x dx = 1 dt$

$$= \int \frac{1}{3t^2 - 4t + 1} dt$$

$$= \int \frac{1}{3 \left(t^2 - \frac{4}{3}t + \frac{1}{3} \right)} dt$$

$$\therefore \left\{ \left(\frac{1}{2} \text{coefficient of } t \right)^2 \right.$$

$$= \left. \left(\frac{1}{2} \left(-\frac{4}{3} \right) \right)^2 = \left(-\frac{2}{3} \right)^2 = \frac{4}{9} \right\}$$

$$I = \frac{1}{3} \int \frac{1}{t^2 - \frac{4}{3}t + \frac{4}{9} - \frac{4}{9} + \frac{1}{3}} dt$$

$$= \frac{1}{3} \int \frac{1}{\left(t^2 - \frac{4}{3}t + \frac{4}{9} \right) - \frac{1}{9}} dt$$

$$= \frac{1}{3} \int \frac{1}{\left(t - \frac{2}{3} \right)^2 - \left(\frac{1}{3} \right)^2} dt$$

$$= \frac{1}{3} \frac{1}{2 \left(\frac{1}{3} \right)} \log \left(\frac{\left(t - \frac{2}{3} \right) - \frac{1}{3}}{\left(t - \frac{2}{3} \right) + \frac{1}{3}} \right) + c$$

$$= \frac{1}{2} \log \left(\frac{3t - 3}{3t - 1} \right) + c$$

$$= \frac{1}{2} \log \left(\frac{3 \sin^2 x - 3}{3 \sin^2 x - 1} \right) + c$$

$$\therefore \int \frac{\sin 2x}{3 \sin^4 x - 4 \sin^2 x + 1} dx$$

$$= \frac{1}{2} \log \left(\frac{3 \sin^2 x - 3}{3 \sin^2 x - 1} \right) + c$$

9. $\int \frac{e^{\frac{x}{2}}}{\sqrt{e^{-x} - e^x}} dx$

Solution :

$$I = \int \frac{\sqrt{e^x}}{\sqrt{\frac{1}{e^x} - e^x}} dx$$

$$= \int \frac{\sqrt{e^x}}{\sqrt{\frac{1 - (e^x)^2}{e^x}}} dx$$

$$= \int \frac{\sqrt{e^x}}{\frac{\sqrt{1 - (e^x)^2}}{\sqrt{e^x}}} dx$$

$$= \int \frac{\sqrt{e^x} \cdot \sqrt{e^x}}{\sqrt{1 - (e^x)^2}} dx$$

$$= \int \frac{e^x}{\sqrt{1 - (e^x)^2}} dx$$

put $e^x = t$

$$\therefore e^x dx = 1 dt$$

$$I = \int \frac{1}{\sqrt{1 - t^2}} dt$$

$$= \sin^{-1}(t) + c$$

$$= \sin^{-1}(e^x) + c$$

$$\therefore \int \frac{e^{\frac{x}{2}}}{\sqrt{e^{-x} - e^x}} dx = \sin^{-1}(e^x) + c$$

10. $\int (\sqrt{\tan x} + \sqrt{\cot x}) dx$

Solution : $I = \int \left(\sqrt{\tan x} + \frac{1}{\sqrt{\tan x}} \right) dx$

$$= \int \frac{\tan x + 1}{\sqrt{\tan x}} dx$$

put $\sqrt{\tan x} = t \quad \therefore \tan x = t^2 \therefore x = \tan^{-1} t^2$

$$\therefore 1 dx = \frac{1}{1 + (t^2)^2} 2t dt$$

$$\therefore \sec^2 x dx = 2t dt$$

$$\therefore dx = \frac{2t}{\sec^2 x} dx = \frac{2t}{1 + \tan^2 x} dx = \frac{2t}{1 + t^4} dt$$

$$= \int \frac{t^2 + 1}{t} \cdot \frac{2t}{1 + t^4} dt = 2 \int \frac{t^2 + 1}{t^4 + 1} dt$$

$$= 2 \int \frac{1 + \frac{1}{t^2}}{t^2 + \frac{1}{t^2}} dt = 2 \int \frac{\left(1 + \frac{1}{t^2}\right)}{\left(t - \frac{1}{t}\right)^2 + 2} dt$$

put $t - \frac{1}{t} = u \quad \therefore \left[\frac{d}{dt} \left(t - \frac{1}{t} \right) = 1 + \frac{1}{t^2} \right]$

$$\therefore \left(t - \left(-\frac{1}{t^2} \right) \right) dt = 1 du$$

$$\therefore \left(1 + \frac{1}{t^2} \right) dt = 1 du$$

$$I = 2 \int \frac{1}{u^2 + 2} du$$

$$= 2 \int \frac{1}{u^2 + (\sqrt{2})^2} du$$

$$= 2 \cdot \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{u}{\sqrt{2}} \right) + c$$

$$= \sqrt{2} \tan^{-1} \left(\frac{t - \frac{1}{t}}{\sqrt{2}} \right) + c$$

$$= \sqrt{2} \tan^{-1} \left(\frac{t^2 - 1}{\sqrt{2} t} \right) + c$$

$$= \sqrt{2} \tan^{-1} \left(\frac{\tan x - 1}{\sqrt{2} \cdot \sqrt{\tan x}} \right) + c$$

11. $\int \frac{1}{5 - 4 \cos x} dx$

Solution : put $\tan \frac{x}{2} = t$

$$\therefore dx = \frac{2}{1 + t^2} dt \quad \text{and} \quad \cos x = \frac{1 - t^2}{1 + t^2}$$

$$I = \int \frac{1 \left(\frac{2}{1 + t^2} \right)}{5 - 4 \left(\frac{1 - t^2}{1 + t^2} \right)} dt$$

$$= \int \frac{\frac{2}{1 + t^2}}{\frac{5(1 + t^2) - 4(1 - t^2)}{1 + t^2}} dt$$

$$= \int \frac{2}{5 - 5t^2 - 4 + 4t^2} dt$$

$$= \int \frac{2}{9t^2 + 1} dt$$

$$= \int \frac{1}{9 \left(t^2 + \frac{1}{9} \right)} dt$$

$$= \frac{2}{9} \int \frac{1}{t^2 + \left(\frac{1}{3} \right)^2} dt$$

$$= \frac{2}{9} \cdot \frac{1}{\left(\frac{1}{3} \right)} \tan^{-1} \left(\frac{t}{\left(\frac{1}{3} \right)} \right) + c$$

$$= \frac{2}{3} \tan^{-1} (3t) + c$$

$$= \frac{2}{3} \tan^{-1} \left(2 \tan \frac{x}{2} \right) + c$$

$$\therefore \int \frac{1}{5 - 4 \cos x} dx = \frac{2}{3} \tan^{-1} \left(2 \tan \frac{x}{2} \right) + c$$

12. $\int \frac{1}{2-3 \sin 2x} dx$

Solution : put $\tan x = t$

$$\therefore dx = \frac{1}{1+t^2} dt \quad \text{and} \quad \sin 2x = \frac{2}{1+t^2}$$

$$\begin{aligned} I &= \int \frac{1 \left(\frac{1}{1+t^2} \right)}{2-3 \left(\frac{2}{1+t^2} \right)} dt \\ &= \int \frac{\frac{1}{1+t^2}}{\frac{2(1+t^2)-3(2t)}{1+t^2}} dt \\ &= \int \frac{1}{2+2t^2-6t} dt = \int \frac{1}{2(t^2-3t+1)} dt \\ \therefore \left\{ \left(\frac{1}{2} \text{ coefficient of } t \right)^2 \right. \\ &= \left. \left(\frac{1}{2}(-3) \right)^2 = \left(-\frac{3}{2} \right)^2 = \frac{9}{4} \right\} \\ &= \frac{1}{2} \int \frac{1}{t^2-3t+\frac{9}{4}-\frac{9}{4}+1} dt \\ &= \frac{1}{2} \int \frac{1}{\left(t^2-3t+\frac{9}{4} \right) - \frac{5}{4}} dt \\ &= \frac{1}{2} \int \frac{1}{\left(t-\frac{3}{2} \right)^2 - \left(\frac{\sqrt{5}}{2} \right)^2} dt \\ &= \frac{1}{2} \frac{1}{2 \left(\frac{\sqrt{5}}{2} \right)} \log \left(\frac{\left(t-\frac{3}{2} \right) - \frac{\sqrt{5}}{2}}{\left(t-\frac{3}{2} \right) + \frac{\sqrt{5}}{2}} \right) + c \\ &= \frac{1}{2\sqrt{5}} \log \left(\frac{2t-3-\sqrt{5}}{2t-3+\sqrt{5}} \right) + c \\ &= \frac{1}{2\sqrt{5}} \log \left(\frac{2 \tan x - 3 - \sqrt{5}}{2 \tan x - 3 + \sqrt{5}} \right) + c \end{aligned}$$

13. $\int \frac{1}{3-2 \sin x + 5 \cos x} dx$

Solution : put $\tan \frac{x}{2} = t$

$$\therefore dx = \frac{2}{1+t^2}$$

$$\therefore \sin x = \frac{2}{1+t^2} dt \quad \text{and} \quad \cos x = \frac{1-t^2}{1+t^2}$$

$$\begin{aligned} I &= \int \frac{1 \left(\frac{2}{1+t^2} \right)}{3-2 \left(\frac{2}{1+t^2} \right) + 5 \left(\frac{1-t^2}{1+t^2} \right)} dt \\ &= \int \frac{\frac{2}{1+t^2}}{\frac{3(1+t^2)-2(2t)+5(1-t^2)}{1+t^2}} dt \\ &= \int \frac{2}{3+3t^2-4t+5-5t^2} dt \\ &= \int \frac{2}{8-4t-2t^2} dt \\ &= \int \frac{1}{4-2t-t^2} dt \\ &= \int \frac{1}{4-(t^2+2t)} dt \\ &= \int \frac{1}{4-(t^2+2t+1-1)} dt \\ &= \int \frac{1}{5-(t^2+2t+1)} dt \\ &= \int \frac{1}{(\sqrt{5})^2-(t+1)^2} dt \\ &= \frac{1}{2(\sqrt{5})} \log \left(\frac{\sqrt{5}+(t+1)}{\sqrt{5}-(t+1)} \right) + c \\ &= \frac{1}{2\sqrt{5}} \log \left(\frac{\sqrt{5}+1+\tan \frac{x}{2}}{\sqrt{5}-1-\tan \frac{x}{2}} \right) + c \end{aligned}$$

Activity : 14. $\int \frac{1}{\sin x - \sqrt{3} \cos x} dx$

Solution : put $\tan \frac{x}{2} = t \quad \therefore dx = \dots\dots\dots$

$\therefore \sin x = \dots\dots\dots$ and $\cos x = \dots\dots\dots$

$$\begin{aligned}
 I &= \int \frac{1 \left(\frac{2}{1+t^2} \right)}{\dots\dots\dots + \sqrt{3} \dots\dots\dots} dt \\
 &= \int \frac{\frac{2}{1+t^2}}{\dots\dots\dots} dt \\
 &= \int \frac{2}{\dots\dots\dots} dt \\
 &= \int \frac{2}{\sqrt{3} \left(1 - \left(t^2 - \frac{2}{\sqrt{3}} t \right) \right)} dt \\
 &= \frac{2}{\sqrt{3}} \int \frac{1}{1 - \left(t^2 - \frac{2}{\sqrt{3}} t + \frac{1}{3} - \frac{1}{3} \right)} dt \\
 &= \frac{2}{\sqrt{3}} \int \frac{1}{1 - \left(t^2 - \frac{2}{\sqrt{3}} t + \frac{1}{3} \right) + \frac{1}{3}} dt \\
 &= \frac{2}{\sqrt{3}} \int \frac{1}{\left(\dots\dots \right) - \left(t - \frac{1}{\sqrt{3}} \right)^2} dt \\
 &= \frac{2}{\sqrt{3}} \int \frac{1}{(\dots)^2 - (\dots)^2} dt \\
 &= \frac{2}{\sqrt{3}} \frac{1}{2(\dots)} \cdot \log \left(\frac{\dots}{\dots} \right) + c \\
 &= \dots \log \left(\frac{\dots}{\dots} \right) + c \\
 &= \frac{1}{2} \log \left(\frac{1 + \sqrt{3} \tan \frac{x}{2}}{3 - \sqrt{3} \tan \frac{x}{2}} \right) + c
 \end{aligned}$$

Alternative method :

14. $\int \frac{1}{\sin x - \sqrt{3} \cos x} dx$

Solution : For any two positive numbers a and b , we can find an angle θ , such that

$$\therefore \sin \theta = \frac{a}{\sqrt{a^2 + b^2}} \quad \text{and} \quad \cos \theta = \frac{b}{\sqrt{a^2 + b^2}}$$

Using this we express $\sin x - \sqrt{3} \cos x$

$$= \sqrt{1+3} (\cos \theta \cdot \sin x - \sin \theta \cdot \cos x)$$

$$= 2 \sin (x - \theta)$$

$$= 2 \sin \left(x - \frac{\pi}{3} \right)$$

$$\therefore I = \int \frac{1}{2 \cdot \sin \left(x - \frac{\pi}{3} \right)} dx$$

$$= \frac{1}{2} \int \operatorname{cosec} \left(x - \frac{\pi}{3} \right) dx$$

$$= \frac{1}{2} \log \left(\operatorname{cosec} \left(x - \frac{\pi}{3} \right) - \cot \left(x - \frac{\pi}{3} \right) \right) + c$$

$$= \frac{1}{2} \log \left(\tan \left(\frac{x}{2} + \frac{\pi}{6} \right) \right) + c$$

15. $\int \frac{1}{3 + 2 \sin^2 x + 5 \cos^2 x} dx$

Solution : Divide Numerator and Denominator by $\cos^2 x$

$$\begin{aligned} I &= \int \frac{\frac{1}{\cos^2 x}}{\frac{3 + 2 \sin^2 x + 5 \cos^2 x}{\cos^2 x}} dx \\ &= \int \frac{\sec^2 x}{3 \sec^2 x + 2 \tan^2 x + 5} dx \\ &= \int \frac{\sec^2 x}{3(1 + \tan^2 x) + 2 \tan^2 x + 5} dx \\ &= \int \frac{\sec^2 x}{5 \tan^2 x + 8} dx \\ &= \frac{1}{5} \int \frac{\sec^2 x}{\tan^2 x + \frac{8}{5}} dx \end{aligned}$$

put $\tan x = t \therefore \sec^2 x dx = 1 dt$

$$\begin{aligned} I &= \frac{1}{5} \int \frac{1}{t^2 + \frac{8}{5}} dt \\ &= \frac{1}{5} \int \frac{1}{t^2 + \left(\frac{\sqrt{8}}{\sqrt{5}}\right)^2} dt \\ &= \frac{1}{5} \frac{1}{\frac{\sqrt{8}}{\sqrt{5}}} \tan^{-1} \left(\frac{t}{\frac{\sqrt{8}}{\sqrt{5}}} \right) + c \\ &= \frac{1}{\sqrt{5}} \frac{1}{2\sqrt{2}} \tan^{-1} \left(\frac{\sqrt{5}t}{2\sqrt{2}} \right) + c \\ &= \frac{1}{2\sqrt{10}} \tan^{-1} \left(\frac{\sqrt{5} \tan x}{2\sqrt{2}} \right) + c \\ \therefore \int \frac{1}{3 + 2 \sin^2 x + 5 \cos^2 x} dx &= \frac{1}{2\sqrt{10}} \tan^{-1} \left(\frac{\sqrt{5} \tan x}{2\sqrt{2}} \right) + c \end{aligned}$$

16. $\int \frac{\cos \theta}{\cos 3\theta} d\theta$

Solution : $I = \int \frac{\cos \theta}{4 \cos^3 \theta - 3 \cos \theta} d\theta$
 $= \int \frac{1}{4 \cos^2 \theta - 3} d\theta$

Divide Numerator and Denominator by $\cos^2 \theta$

$$\begin{aligned} I &= \int \frac{\frac{1}{\cos^2 \theta}}{\frac{4 \cos^2 \theta - 3}{\cos^2 \theta}} d\theta \\ &= \int \frac{\sec^2 \theta}{4 - 3 \sec^2 \theta} d\theta \\ &= \int \frac{\sec^2 \theta}{4 - 3(1 + \tan^2 \theta)} d\theta \\ &= \int \frac{\sec^2 \theta}{1 - 3 \tan^2 \theta} d\theta \end{aligned}$$

put $\tan \theta = t \therefore \sec^2 \theta d\theta = 1 dt$

$$\begin{aligned} I &= \int \frac{1}{1 - 3t^2} dt \\ &= \frac{1}{3} \int \frac{1}{\frac{1}{3} - t^2} dt \\ &= \frac{1}{3} \int \frac{1}{\left(\frac{1}{\sqrt{3}}\right)^2 - t^2} dt \\ &= \frac{1}{3} \frac{1}{2\left(\frac{1}{\sqrt{3}}\right)} \log \left(\frac{\frac{1}{\sqrt{3}} + t}{\frac{1}{\sqrt{3}} - t} \right) + c \\ &= \frac{1}{2\sqrt{3}} \log \left(\frac{1 + \sqrt{3}t}{1 - \sqrt{3}t} \right) + c \\ &= \frac{1}{2\sqrt{3}} \log \left(\frac{1 + \sqrt{3} \tan \theta}{1 - \sqrt{3} \tan \theta} \right) + c \\ \therefore \int \frac{\cos \theta}{\cos 3\theta} d\theta &= \frac{1}{2\sqrt{3}} \log \left(\frac{1 + \sqrt{3} \tan \theta}{1 - \sqrt{3} \tan \theta} \right) + c \end{aligned}$$

EXERCISE 3.2 (B)

I. Evaluate the following :

1. $\int \frac{1}{4x^2 - 3} dx$

2. $\int \frac{1}{25 - 9x^2} dx$

3. $\int \frac{1}{7 + 2x^2} dx$

4. $\int \frac{1}{\sqrt{3x^2 + 8}} dx$

5. $\int \frac{1}{\sqrt{11 - 4x^2}} dx$

6. $\int \frac{1}{\sqrt{2x^2 - 5}} dx$

7. $\int \sqrt{\frac{9+x}{9-x}} dx$

8. $\int \sqrt{\frac{2+x}{2-x}} dx$

9. $\int \sqrt{\frac{10+x}{10-x}} dx$

10. $\int \frac{1}{x^2 + 8x + 12} dx$

11. $\int \frac{1}{1 + x - x^2} dx$

12. $\int \frac{1}{4x^2 - 20x + 17} dx$

13. $\int \frac{1}{5 - 4x - 3x^2} dx$

14. $\int \frac{1}{\sqrt{3x^2 + 5x + 7}} dx$

15. $\int \frac{1}{\sqrt{x^2 + 8x - 20}} dx$

16. $\int \frac{1}{\sqrt{8 - 3x + 2x^2}} dx$

17. $\int \frac{1}{\sqrt{(x-3)(x+2)}} dx$

18. $\int \frac{1}{4 + 3 \cos^2 x} dx$

19. $\int \frac{1}{\cos 2x + 3 \sin^2 x} dx$

20. $\int \frac{\sin x}{\sin 3x} dx$

II. Integrate the following functions w. r. t. x :

1. $\int \frac{1}{3 + 2 \sin x} dx$

2. $\int \frac{1}{4 - 5 \cos x} dx$

3. $\int \frac{1}{2 + \cos x - \sin x} dx$

4. $\int \frac{1}{3 + 2 \sin x - \cos x} dx$

5. $\int \frac{1}{3 - 2 \cos 2x} dx$

6. $\int \frac{1}{2 \sin 2x - 3} dx$

7. $\int \frac{1}{3 + 2 \sin 2x + 4 \cos 2x} dx$

8. $\int \frac{1}{\cos x - \sin x} dx$

9. $\int \frac{1}{\cos x - \sqrt{3} \sin x} dx$

3.2.6 Integral of the form $\int \frac{px+q}{ax^2+bx+c} dx$ and $\int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$

The integral of the form $\int \frac{px+q}{ax^2+bx+c} dx$ is evaluated by expressing the integral in the form

$$\int \frac{A \cdot \frac{d}{dx}(ax^2+bx+c)}{ax^2+bx+c} dx + \int \frac{B}{ax^2+bx+c} dx \text{ for some constants } A \text{ and } B.$$

The numerator, $px+q = A \frac{d}{dx}(ax^2+bx+c) + B$

$$\text{i.e. } N = A \frac{d}{dx} D + B$$

The first integral is evaluated by putting $ax^2+bx+c = t$

The Second integral is evaluated by expressing the integrand in the form either

$$\frac{1}{A^2+t^2} \text{ or } \frac{1}{t^2-A^2} \text{ or } \frac{1}{A^2-t^2} \text{ and applying the methods discussed previously.}$$

The integral of the form $\int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$ is evaluated by expressing the integral in the form

$$\int \frac{A \frac{d}{dx}(ax^2+bx+c)}{\sqrt{ax^2+bx+c}} dx + \int \frac{B}{\sqrt{ax^2+bx+c}} dx \text{ for constants } A \text{ and } B.$$

The numerator, $px+q = A \frac{d}{dx}(ax^2+bx+c) + B$

The first integral is evaluated by putting $ax^2+bx+c = t$

The second integral is evaluated by expressing the integrand in the form either

$$\frac{1}{\sqrt{A^2+t^2}} \text{ or } \frac{1}{\sqrt{t^2-A^2}} \text{ or } \frac{1}{\sqrt{A^2-t^2}} \text{ and applying the methods which discussed previously.}$$



SOLVED EXAMPLES

1. $\int \frac{2x-3}{3x^2+4x+5} dx$

Solution : $2x-3 = A \frac{d}{dx} (3x^2+4x+5) + B$

$$2x-3 = A(6x+4) + B$$

$$= (6A)x + (4A+B)$$

comparing the sides/ the co-efficients of like variables and constants

$$6A = 2 \text{ and } 4A + B = -3$$

$$\Rightarrow A = \frac{1}{3} \text{ and } B = -\frac{13}{3}$$

$$= \int \frac{\frac{1}{3} \cdot \frac{d}{dx} (3x^2+4x+5) + \left(-\frac{13}{3}\right)}{3x^2+4x+5} dx$$

$$= \frac{1}{3} \int \frac{\frac{d}{dx} (3x^2+4x+5)}{3x^2+4x+5} dx - \frac{13}{3} \int \frac{1}{3x^2+4x+5} dx$$

$$= \frac{1}{3} \int \frac{6x+4}{3x^2+4x+5} dx - \frac{13}{3} \int \frac{1}{3x^2+4x+5} dx$$

$$= I_1 - I_2 \quad \dots\dots (i)$$

$$\therefore I_1 = \frac{1}{3} \int \frac{6x+4}{3x^2+4x+5} dx$$

put $3x^2+4x+5 = t$

$$\therefore (6x+4) dx = 1 dt$$

$$I_1 = \frac{1}{3} \int \frac{1}{t} dt$$

$$= \frac{1}{3} \log(t) + c_1$$

$$= \frac{1}{3} \log(3x^2+4x+5) + c_1 \quad \dots\dots (ii)$$

$$\therefore I_2 = \frac{13}{3} \int \frac{1}{3x^2+4x+5} dx$$

$$= \frac{13}{3} \frac{1}{3} \int \frac{1}{x^2 + \frac{4}{3}x + \frac{5}{3}} dx$$

$$\therefore \left\{ \left(\frac{1}{2} \text{ coefficient of } t \right)^2 \right. \\ \left. = \left(\frac{1}{2} \left(\frac{4}{3} \right) \right)^2 = \left(\frac{2}{3} \right)^2 = \frac{4}{9} \right\}$$

$$I_2 = \frac{13}{9} \int \frac{1}{x^2 + \frac{4}{3}x + \frac{4}{9} - \frac{4}{9} + \frac{5}{3}} dx$$

$$= \frac{13}{9} \int \frac{1}{x^2 + \frac{4}{3}x + \frac{4}{9} + \frac{11}{9}} dx$$

$$= \frac{13}{9} \int \frac{1}{\left(x + \frac{2}{3}\right)^2 + \left(\frac{\sqrt{11}}{3}\right)^2} dx$$

$$\therefore \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$$

$$= \frac{13}{9} \frac{1}{\frac{\sqrt{11}}{3}} \tan^{-1}\left(\frac{x + \frac{2}{3}}{\frac{\sqrt{11}}{3}}\right) + c_1$$

$$I_2 = \frac{13}{3\sqrt{11}} \tan^{-1}\left(\frac{3x+2}{\sqrt{11}}\right) + c_2 \quad \dots\dots (iii)$$

thus, from (i), (ii) and (iii)

$$\therefore \int \frac{2x-3}{3x^2+4x+5} dx \\ = \frac{1}{3} \log(3x^2+4x+5) - \frac{13}{3\sqrt{11}} \tan^{-1}\left(\frac{3x+2}{\sqrt{11}}\right) + c \\ (\because c_1 + c_2 = c)$$

2. $\int \sqrt{\frac{x-5}{x-7}} dx$

Solution : $I = \int \sqrt{\frac{(x-5)(x-5)}{(x-7)(x-5)}} dx = \int \sqrt{\frac{(x-5)^2}{x^2-12x+35}} dx$

$\therefore x-5 = A \frac{d}{dx}(x^2-12x+35) + B$

$x-5 = A(2x-12) + B$

$= (2A)x + (-12A+B)$

comparing, the co-efficients of like variables and constants

$2A = 1 \text{ and } -12A + B = -5$

$\Rightarrow A = \frac{1}{2} \text{ and } B = 1$

$$\begin{aligned} I &= \int \frac{\frac{1}{2} \frac{d}{dx}(x^2-12x+35) + (1)}{\sqrt{x^2-12x+35}} dx \\ &= \frac{1}{2} \int \frac{\frac{d}{dx}(x^2-12x+35)}{\sqrt{x^2-12x+35}} dx + \int \frac{1}{\sqrt{x^2-12x+35}} dx \\ &= I_1 + I_2 \quad \dots\dots (i) \end{aligned}$$

$\therefore I_1 = \frac{1}{2} \int \frac{2x-12}{\sqrt{x^2-12x+35}} dx$

put $x^2-12x+35 = t$

$\therefore (2x-12) dx = 1 dt$

$I_1 = \frac{1}{2} \int \frac{1}{\sqrt{t}} dt$

$= \int \frac{1}{2\sqrt{t}} dt$

$= \sqrt{t} + c_1$

$= \sqrt{x^2-12x+35} + c_1 \quad \dots\dots (ii)$

$\therefore I_2 = \int \frac{1}{\sqrt{x^2-12x+35}} dx$

$= \int \frac{1}{\sqrt{x^2-12x+36-1}} dx$

$= \int \frac{1}{\sqrt{(x-6)^2-(1)^2}} dx$

$\therefore \int \frac{1}{\sqrt{x^2-a^2}} dx = \log(x + \sqrt{x^2-a^2}) + c$

$I_2 = \log((x-6) + \sqrt{(x-6)^2-1}) + c_2$

$= \log((x-6) + \sqrt{x^2-12x+35}) + c_2$

$\dots\dots (iii)$

Thus, from (i), (ii) and (iii)

$\int \sqrt{\frac{x-5}{x-7}} dx$

$= \sqrt{x^2-12x+35} + \log((x-6) + \sqrt{x^2-12x+35}) + c$

$(c_1 + c_2 = c)$

Activity :

3. $\int \sqrt{\frac{8-x}{x}} dx$

Solution : $= \int \sqrt{\frac{(8-x) \times (\quad)}{x \times (\quad)}} dx = \int \sqrt{\frac{(\quad)^2}{8x-x^2}} dx = \int \frac{(8-x)}{\sqrt{8x-x^2}} dx$

$\therefore 8-x = A \frac{d}{dx}(8x-x^2) + B$

$8-x = A(\dots\dots\dots) + B$

$= (8A+B) - 2Ax$

comparing, the co-efficients of like variables and constants

$8A+B = \dots\dots$ and $-2A = -1$

$\Rightarrow A = \frac{\dots}{\dots}$ and $B = \dots\dots$

$= \int \frac{\frac{1}{2} \frac{d}{dx}(8x-x^2) + (4)}{\sqrt{8x-x^2}} dx$

$= \frac{1}{2} \int \frac{d}{dx}(8x-x^2) dx + 4 \int \frac{1}{\sqrt{8x-x^2}} dx$

$= \frac{1}{2} \int \frac{8-2x}{\sqrt{8x-x^2}} dx + 4 \int \frac{1}{\sqrt{8x-x^2}} dx$

$= I_1 + I_2 \dots\dots (i)$

$\therefore I_1 = \frac{1}{2} \int \frac{8-2x}{\sqrt{8x-x^2}} dx$

put $\dots\dots\dots = t$

$\therefore (\dots\dots\dots) dx = 1 dt$

$= \frac{1}{2} \int \frac{1}{\sqrt{t}} dt$

$= \int \frac{1}{2\sqrt{t}} dt$

$= \sqrt{t} + c_1$

$= \sqrt{8x-x^2} + c_1 \dots\dots (ii)$

$\therefore I_2 = 4 \int \frac{1}{\sqrt{8x-x^2}} dx$

$= 4 \int \frac{1}{\sqrt{-(\dots\dots\dots)}} dx$

$= 4 \int \frac{1}{\sqrt{\dots\dots\dots - (\dots\dots\dots)}} dx$

$= 4 \int \frac{1}{\sqrt{\dots\dots\dots - (x-4)^2}} dx$

$\therefore \int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

$I_2 = 4 \sin^{-1}\left(\frac{x-4}{4}\right) + c_2 \dots\dots (iii)$

thus, from (i), (ii) and (iii)

$\therefore \int \sqrt{\frac{8-x}{x}} dx$

$= \sqrt{8x-x^2} + 4 \sin^{-1}\left(\frac{x-4}{4}\right) + c$

$(\because c_1 + c_2 = c)$

EXERCISE 3.2 (C)

I. Evaluate :

1. $\int \frac{3x+4}{x^2+6x+5} dx$
2. $\int \frac{2x+1}{x^2+4x-5} dx$
3. $\int \frac{2x+3}{2x^2+3x-1} dx$
4. $\int \frac{3x+4}{\sqrt{2x^2+2x+1}} dx$
5. $\int \frac{7x+3}{\sqrt{3+2x-x^2}} dx$
6. $\int \sqrt{\frac{x-7}{x-9}} dx$
7. $\int \sqrt{\frac{9-x}{x}} dx$
8. $\int \frac{3 \cos x}{4 \sin^2 x + 4 \sin x - 1} dx$
9. $\int \sqrt{\frac{e^{3x} - e^{2x}}{e^x + 1}} dx$

3.3 Integration by parts :

This method is useful when the integrand is expressed as a product of two different types of functions; one of which can be differentiated and the other can be integrated conveniently.

The following theorem gives the rule of integration by parts.

3.3.1 Theorem : If u and v are two differentiable functions of x then

$$\int u v dx = u \int v dx - \int \left(\frac{d}{dx} u \right) (\int v dx) dx$$

Proof : Let $\int v dx = w \quad \dots (i) \Rightarrow v = \frac{dw}{dx} \quad \dots (ii)$

$$\begin{aligned} \text{Consider, } \frac{d}{dx} (u w) &= u \frac{d}{dx} w + w \frac{d}{dx} u \\ &= u v + w \frac{du}{dx} \end{aligned}$$

By definition of integration

$$\begin{aligned} u w &= \int \left[u v + w \frac{du}{dx} \right] dx \\ &= \int u v dx + \int w \frac{du}{dx} dx \\ &= \int u v dx + \int \frac{du}{dx} w dx \end{aligned}$$

$$\therefore u \int v dx = \int u v dx + \int \frac{du}{dx} \int v dx dx$$

$$\therefore \int u v dx = u \int v dx - \int \left(\frac{d}{dx} u \right) (\int v dx) dx$$

For example : $\int x \cdot e^x dx = x \int e^x dx - \int \left(\frac{d(x)}{dx} \int e^x dx \right) dx$

$$= x e^x - \int (1) e^x dx$$

$$= x e^x - \int e^x dx$$

$$= x e^x - e^x + c$$

now let us reverse the choice of u and v

$$\therefore \int e^x \cdot x dx = e^x \int x^1 dx - \int \frac{d}{dx} e^x \left(\int x dx \right) dx$$

$$= e^x \frac{x^2}{2} - \int e^x \frac{x^2}{2} dx$$

$$= \frac{1}{2} e^x x^2 - \frac{1}{2} \int e^x x^2 dx$$

We arrive at an integral $\int e^x x^2 dx$ which is more difficult, but it helps to get $\int e^x x^2 dx$

Thus it is essential to make a proper choice of the first function and the second function. The first function to be selected will be the one, which comes first in the order of **L I A T E**.

L **Logarithmic function.**
I **Inverse trigonometric function.**
A **Algebraic function.**
T **Trigonometric function.**
E **Exponential function.**

For example : $\int \sin x \cdot x dx$

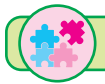
$$= \int x \sin x dx \quad \dots \dots \text{by LIATE}$$

$$= x \int \sin x dx - \int \frac{d}{dx} x \left(\int \sin x dx \right) dx$$

$$= x (-\cos x) - \int (1) (-\cos x) dx$$

$$= -x \cos x + \int \cos x dx$$

$$= -x \cos x + \sin x + c$$



SOLVED EXAMPLES

1. $\int x^2 5^x dx$

Solution : I = $x^2 \int 5^x dx - \int \frac{d}{dx} x^2 \left(\int 5^x dx \right) dx$

$$= x^2 5^x \frac{1}{\log 5} - \int 2x 5^x \frac{1}{\log 5} dx$$

$$= \frac{1}{\log 5} x^2 5^x - \frac{2}{\log 5} \left\{ x \int 5^x dx - \int \frac{d}{dx} x \left(\int 5^x dx \right) dx \right\}$$

$$= \frac{1}{\log 5} x^2 5^x - \frac{2}{\log 5} \left\{ x 5^x \frac{1}{\log 5} - \int (1) \left(5^x \frac{1}{\log 5} \right) dx \right\}$$

$$= \frac{1}{\log 5} x^2 5^x - \frac{2}{\log 5} \left\{ \frac{1}{\log 5} x 5^x - \int \frac{1}{\log 5} 5^x dx \right\}$$

$$= \frac{1}{\log 5} x^2 5^x - \frac{2}{\log 5} \left\{ \frac{1}{\log 5} x 5^x - \frac{1}{\log 5} 5^x \frac{1}{\log 5} \right\} + c$$

$$= \frac{1}{\log 5} x^2 5^x - \frac{2}{(\log 5)^2} x 5^x + \frac{2}{(\log 5)^3} 5^x + c$$

$$\therefore \int x^2 5^x dx = \frac{5^x}{\log 5} \left\{ x^2 - \frac{2x}{\log 5} + \frac{2}{(\log 5)^2} \right\} + c$$

2. $\int x \tan^{-1} x dx$

Solution : I = $\int (\tan^{-1} x) x dx$ by LIATE

$$= \tan^{-1} x \int x dx - \int \frac{d}{dx} \tan^{-1} x \left(\int x dx \right) dx$$

$$= \tan^{-1} x \frac{x^2}{2} - \int \frac{1}{1+x^2} \frac{x^2}{2} dx$$

$$= \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{1+x^2} dx$$

$$= \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} \int \frac{1+x^2-1}{1+x^2} dx$$

$$= \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} \int \left[1 - \frac{1}{1+x^2} \right] dx$$

$$= \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} [x - \tan^{-1} x] + c$$

$$\therefore \int x \tan^{-1} x dx = \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} x + \frac{1}{2} \tan^{-1} x + c$$

3. $\int \frac{x}{1 - \sin x} dx$

Solution : $I = \int \frac{x}{1 - \sin x} \frac{(1 + \sin x)}{(1 + \sin x)} dx$

$$= \int \frac{x(1 + \sin x)}{1 - \sin^2 x} dx = \int \frac{x(1 + \sin x)}{\cos^2 x} dx = \int x \left(\frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x} \right) dx$$

$$= \int x (\sec^2 x + \sec x \tan x) dx$$

$$= \int x \sec^2 x dx + \int x \sec x \tan x dx$$

$$= \left(x \int \sec^2 x dx - \int \frac{d}{dx} x \left(\int \sec^2 x dx \right) dx \right) + \left(x \int \sec x \tan x dx - \int \frac{d}{dx} x \left(\int \sec x \tan x dx \right) dx \right)$$

$$= x \tan x - \int (1) \tan x dx + x \sec x - \int (1) \sec x dx$$

$$= x \tan x - \log (\sec x) + x \sec x - \log (\sec x + \tan x) + c$$

$$= x (\sec x + \tan x) - \log (\sec x) - \log (\sec x + \tan x) + c$$

$$\therefore \int \frac{x}{1 - \sin x} dx = x (\sec x + \tan x) - \log [(\sec x) (\sec x + \tan x)] + c$$

4. $\int e^{2x} \sin 3x dx$

Solution : $I = \int e^{2x} \sin 3x dx$

Here we use repeated integration by parts.

To evaluate $\int e^{ax} \sin (bx + c) dx$; $\int e^{ax} \cos (bx + c) dx$ any function can be taken as a first function.

$$I = e^{2x} \int \sin 3x dx - \int \frac{d}{dx} e^{2x} \left(\int \sin 3x dx \right) dx$$

$$= e^{2x} \left(-\cos 3x \frac{1}{3} \right) - \int e^{2x} 2 \left(-\cos 3x \frac{1}{3} \right) dx$$

$$= -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{3} \int e^{2x} \cos 3x dx$$

$$= -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{3} \left(e^{2x} \int \cos 3x \cdot dx - \int \frac{d}{dx} e^{2x} \left(\int \cos 3x dx \right) dx \right)$$

$$= -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{3} \left[e^{2x} \left(\sin 3x \frac{1}{3} \right) - \int e^{2x} 2 \left(\sin 3x \frac{1}{3} \right) dx \right]$$

$$= -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{9} e^{2x} \sin 3x - \frac{4}{9} \int e^{2x} \sin 3x dx$$

$$I = -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{9} e^{2x} \sin 3x - \frac{4}{9} I$$

$$I + \frac{4}{9} I = \frac{e^{2x}}{9} [-3 \cos 3x + 2 \sin 3x] + c$$

$$= \frac{e^{2x}}{13} [2 \sin 3x - 3 \cos 3x] + c$$

$$\frac{13}{9} I = \frac{e^{2x}}{9} [2 \sin 3x - 3 \cos 3x] + c$$

$$\therefore \int e^{2x} \sin 3x dx = \frac{e^{2x}}{13} [2 \sin 3x - 3 \cos 3x] + c$$

Activity :

Prove the following results.

$$(i) \int e^{ax} \sin (bx + c) dx = \frac{e^{ax}}{a^2 + b^2} [a \sin (bx + c) - b \cos (bx + c)] + c$$

$$(ii) \int e^{ax} \cos (bx + c) dx = \frac{e^{ax}}{a^2 + b^2} [a \sin (bx + c) + b \cos (bx + c)] + c$$

5. $\int \left[\log (\log x) + \frac{1}{(\log x)^2} \right] dx$

Solution :

$$\begin{aligned} I &= \int \log (\log x) 1 dx + \int \frac{1}{(\log x)^2} dx \\ &= \log (\log x) \int 1 dx - \int \frac{d}{dx} \log (\log x) \int 1 dx + \int \frac{1}{(\log x)^2} dx \\ &= \log (\log x) x - \int \frac{1}{\log x} \cdot \frac{1}{x} (x) dx + \int \frac{1}{(\log x)^2} dx \\ &= \log (\log x) x - \int \frac{1}{\log x} dx + \int \frac{1}{(\log x)^2} dx \\ &= \log (\log x) x - \int (\log x)^{-1} 1 dx + \int \frac{1}{(\log x)^2} dx \\ &= \log (\log x) x - \left\{ (\log x)^{-1} \int 1 dx + \int \frac{d}{dx} (\log x)^{-1} \int 1 dx dx \right\} + \int \frac{1}{(\log x)^2} dx \\ &= \log (\log x) x - \left\{ (\log x)^{-1} x - \int -1 (\log x)^{-2} \frac{1}{x} x dx \right\} + \int \frac{1}{(\log x)^2} dx \\ &= \log (\log x) x - (\log x)^{-1} x - \int (\log x)^{-2} dx + \int \frac{1}{(\log x)^2} dx \\ &= x \log (\log x) - \frac{x}{\log x} - \int \frac{1}{(\log x)^2} dx + \int \frac{1}{(\log x)^2} dx \\ \therefore \int \left[\log (\log x) + \frac{1}{(\log x)^2} \right] dx &= x \log (\log x) - \frac{x}{\log x} + c \end{aligned}$$

Note that :

To evaluate the integrals of type $\int \sin^{-1} x dx$; $\int \tan^{-1} x dx$; $\int \sec^{-1} x dx$; $\int \log x dx$, take the second function (v) to be 1 and then apply integration by parts.

$$\int \sqrt{a^2 - x^2} dx ; \int \sqrt{a^2 + x^2} dx ; \int \sqrt{x^2 - a^2} dx$$

6. $\int \sqrt{a^2 - x^2} \, dx$

Solution : Let $I = \int \sqrt{a^2 - x^2} \, dx$

$$= \sqrt{a^2 - x^2} \int 1 \, dx - \int \frac{d}{dx} \sqrt{a^2 - x^2} \left(\int 1 \, dx \right) dx$$

$$= \sqrt{a^2 - x^2} \, x - \int \frac{1}{2\sqrt{a^2 - x^2}} (-2x) (x) \, dx$$

$$= \sqrt{a^2 - x^2} \, x + \int \frac{x^2}{\sqrt{a^2 - x^2}} \, dx$$

$$= \sqrt{a^2 - x^2} \, x + \int \frac{a^2 - (a^2 - x^2)}{\sqrt{a^2 - x^2}} \, dx$$

$$= \sqrt{a^2 - x^2} \, x + \int \left[\frac{a^2}{\sqrt{a^2 - x^2}} - \frac{(a^2 - x^2)}{\sqrt{a^2 - x^2}} \right] dx$$

$$= x \sqrt{a^2 - x^2} + a^2 \int \frac{1}{\sqrt{a^2 - x^2}} \, dx - \int \sqrt{a^2 - x^2} \, dx$$

$$I = x \sqrt{a^2 - x^2} + a^2 \int \frac{1}{\sqrt{a^2 - x^2}} \, dx - I$$

$$\therefore I + I = x \sqrt{a^2 - x^2} + a^2 \sin^{-1} \left(\frac{x}{a} \right) + c$$

$$\therefore I = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + \frac{c}{2}$$

$$\therefore \int \sqrt{a^2 - x^2} \, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + c_1 \text{ where } c_1 = \frac{c}{2}$$

e.g. $\int \sqrt{9 - x^2} \, dx = \frac{x}{2} \sqrt{9 - x^2} + \frac{9}{2} \sin^{-1} \left(\frac{x}{3} \right) + c$

with reference to the above example solve these :

7. $\int \sqrt{a^2 + x^2} \, dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log \left(x + \sqrt{x^2 + a^2} \right) + c$

8. $\int \sqrt{x^2 - a^2} \, dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log \left(x + \sqrt{x^2 - a^2} \right) + c$

9. $\int x \sin^{-1} x \, dx$

Solution : $I = \int \sin^{-1} x \, x \, dx \dots \dots$ by LIATE

$$\begin{aligned}
 &= \sin^{-1} x \int x \, dx - \int \frac{d}{dx} \sin^{-1} x \left(\int x \, dx \right) dx \\
 &= \sin^{-1} x \frac{x^2}{2} - \int \frac{1}{\sqrt{1-x^2}} \frac{x^2}{2} dx \\
 &= \frac{1}{2} x^2 \sin^{-1} x - \frac{1}{2} \int \frac{x^2}{\sqrt{1-x^2}} dx \\
 &= \frac{1}{2} x^2 \sin^{-1} x - \frac{1}{2} \int \frac{1 - (1-x^2)}{\sqrt{1-x^2}} dx \\
 &= \frac{1}{2} x^2 \sin^{-1} x - \frac{1}{2} \int \left[\frac{1}{\sqrt{1-x^2}} - \frac{(1-x^2)}{\sqrt{1-x^2}} \right] dx \\
 &= \frac{1}{2} x^2 \sin^{-1} x - \frac{1}{2} \int \frac{dx}{\sqrt{1-x^2}} + \frac{1}{2} \int \sqrt{1-x^2} dx \\
 &= \frac{1}{2} x^2 \sin^{-1} x - \frac{1}{2} \sin^{-1} x + \frac{1}{2} \left[\frac{x}{2} \sqrt{1-x^2} + \frac{1}{2} \sin^{-1}(x) \right] + c \\
 &= \frac{1}{2} x^2 \sin^{-1} x + \frac{1}{4} x \sqrt{1-x^2} - \frac{1}{4} \sin^{-1} x + c
 \end{aligned}$$

$$\therefore \int x \sin^{-1} x \, dx = \frac{1}{2} x^2 \sin^{-1} x + \frac{1}{4} x \sqrt{1-x^2} - \frac{1}{4} \sin^{-1} x + c$$

Activity :

10. $\int \cos^{-1} \sqrt{x} \, dx$

Solution : put $\sqrt{x} = t$

$$\therefore x = t^2$$

differentiating w.r.t. x

$$\therefore 1 \, dx = 2t \, dt$$

$$I = \int \cos^{-1} t \, 2t \, dt$$

refer previous (example no. 9) example and solve it.

11. $\int \sqrt{4 + 3x - 2x^2} \, dx$

Solution : $I = \int \sqrt{4 - 2x^2 + 3x} \, dx$

$$= \int \sqrt{4 - 2\left(x^2 - \frac{3}{2}x\right)} \, dx$$

$$= \int \sqrt{2} \sqrt{2 - \left(x^2 - \frac{3}{2}x\right)} \, dx$$

$$\therefore \left\{ \left(\frac{1}{2} \text{ coefficient of } x \right)^2 = \left[\frac{1}{2} \left(-\frac{3}{2} \right) \right]^2 = \left(-\frac{3}{4} \right)^2 = \frac{9}{16} \right\}$$

$$I = \sqrt{2} \int \sqrt{2 - \left(x^2 - \frac{3}{2}x + \frac{9}{16} - \frac{9}{16}\right)} \, dx$$

$$= \sqrt{2} \int \sqrt{2 - \left(x^2 - \frac{3}{2}x + \frac{9}{16}\right) + \frac{9}{16}} \, dx$$

$$= \sqrt{2} \int \sqrt{\left(\frac{\sqrt{41}}{4}\right)^2 - \left(x - \frac{3}{4}\right)^2} \, dx$$

$$\therefore \int \sqrt{a^2 - x^2} \, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$= \sqrt{2} \left[\frac{\left(x - \frac{3}{4}\right)}{2} \sqrt{\left(\frac{\sqrt{41}}{4}\right)^2 - \left(x - \frac{3}{4}\right)^2} + \frac{\left(\frac{\sqrt{41}}{4}\right)^2}{2} \sin^{-1}\left(\frac{x - \frac{3}{4}}{\frac{\sqrt{41}}{4}}\right) \right] + c$$

$$= \sqrt{2} \left[\frac{4x - 3}{8} \sqrt{2 + \frac{3}{2}x - x^2} + \frac{41}{32} \sin^{-1}\left(\frac{4x - 3}{\sqrt{41}}\right) \right] + c$$

$$\therefore \int \sqrt{4 + 3x - 2x^2} \, dx = \frac{4x - 3}{8} \sqrt{4 + 3x - 2x^2} + \frac{41}{16\sqrt{2}} \sin^{-1}\left(\frac{4x - 3}{\sqrt{41}}\right) + c$$

Note that :

3.3.2 :

To evaluate the integral of type $\int (px + q) \sqrt{ax^2 + bx + c} \, dx$

we express the term $px + q = A \frac{d}{dx}(ax^2 + bx + c) + B \dots$ for constants A, B .

Then the integral will be evaluated by the usual known methods.

3.3.3 Integral of the type $\int e^x [f(x) + f'(x)] dx = e^x \cdot f(x) + c$

Let $e^x f(x) = t$

Differentiating w. r. t. x

$$\left[e^x [f'(x) + f(x)] \right] = \frac{dt}{dx}$$

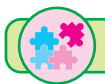
$$e^x [f(x) + f'(x)] = \frac{dt}{dx}$$

By definition of integration,

$$\therefore \int e^x [f(x) + f'(x)] dx = t + c$$

$$\therefore \int e^x [f(x) + f'(x)] dx = e^x \cdot f(x) + c$$

e.g. $\int e^x [\tan x + \sec^2 x] dx = e^x \cdot \tan x + c$
 $\left(\because \frac{d}{dx} \tan x = \sec^2 x \right)$



SOLVED EXAMPLES

1. $\int e^x \left(\frac{2 + \sin 2x}{1 + \cos 2x} \right) dx$

Solution :

$$\begin{aligned} I &= \int e^x \left(\frac{2 + 2 \sin x \cdot \cos x}{2 \cdot \cos^2 x} \right) dx \\ &= \int e^x \left(\frac{1}{\cos^2 x} + \frac{\sin x \cos x}{\cos^2 x} \right) dx \\ &= \int e^x [\sec^2 x + \tan x] dx \\ &= \int e^x [\tan x + \sec^2 x] dx \\ \therefore f(x) = \tan x &\Rightarrow f'(x) = \sec^2 x \\ \therefore \int e^x [f(x) + f'(x)] dx &= e^x \cdot f(x) + c \\ I &= e^x \cdot \tan x + c \\ \therefore \int e^x \left(\frac{2 + \sin 2x}{1 + \cos 2x} \right) dx &= e^x \cdot \tan x + c \end{aligned}$$

2. $\int e^x \left[\frac{x+2}{(x+3)^2} \right] dx$

Solution :

$$\begin{aligned} I &= \int e^x \left[\frac{x+3-1}{(x+3)^2} \right] dx \\ &= \int e^x \left[\frac{x+3}{(x+3)^2} + \frac{-1}{(x+3)^2} \right] dx \\ &= \int e^x \left[\frac{1}{x+3} + \frac{-1}{(x+3)^2} \right] dx \\ \therefore f(x) = \frac{1}{x+3} &\Rightarrow f'(x) = \frac{-1}{(x+3)^2} \\ \therefore \int e^x [f(x) + f'(x)] \cdot dx &= e^x \cdot f(x) + c \\ &= e^x \cdot \left(\frac{1}{x+3} \right) + c \\ &= \frac{e^x}{x+3} + c \\ \therefore \int e^x \left[\frac{x+2}{(x+3)^2} \right] dx &= \frac{e^x}{x+3} + c \end{aligned}$$

$$3. \int e^{\tan^{-1} x} \left(\frac{1+x+x^2}{1+x^2} \right) dx$$

Solution : put $\tan^{-1} x = t$

$$\therefore x = \tan t$$

differentiating w. r. t. x

$$\therefore \frac{1}{1+x^2} dx = 1 dt$$

$$\begin{aligned} I &= \int e^t [1 + \tan t + \tan^2 t] dt \\ &= \int e^t [\tan t + (1 + \tan^2 t)] dt \\ &= \int e^t [\tan t + \sec^2 t] dt \end{aligned}$$

$$\text{Here } f(t) = \tan t$$

$$\Rightarrow f'(t) = \sec^2 t$$

$$\begin{aligned} I &= e^t \cdot f(t) + c \\ &= e^t \cdot \tan t + c \\ &= e^{\tan^{-1} x} \cdot x + c \end{aligned}$$

$$\therefore \int e^{\tan^{-1} x} \cdot \left(\frac{1+x+x^2}{1+x^2} \right) dx = e^{\tan^{-1} x} \cdot x + c$$

$$4. \int \frac{(x^2+1) \cdot e^x}{(x+1)^2} dx$$

Solution :

$$\begin{aligned} I &= \int e^x \left[\frac{x^2+1}{(x+1)^2} \right] dx \\ &= \int e^x \left[\frac{x^2-1+2}{(x+1)^2} \right] dx \\ &= \int e^x \left[\frac{x^2-1}{(x+1)^2} + \frac{2}{(x+1)^2} \right] dx \\ &= \int e^x \left[\frac{x-1}{x+1} + \frac{2}{(x+1)^2} \right] dx \end{aligned}$$

$$\text{Here } f(x) = \frac{x-1}{x+1}$$

$$\Rightarrow f'(x) = \frac{(x+1)(1) - (x-1)(1)}{(x+1)^2} = \frac{2}{(x+1)^2}$$

$$\therefore \int [f(x) + f'(x)] dx = e^x f(x) + c$$

$$I = e^x \left(\frac{x-1}{x+1} \right) + c$$

$$\therefore \int \frac{(x^2+1) e^x}{(x+1)^2} dx = e^x \left(\frac{x-1}{x+1} \right) + c$$

EXERCISE 3.3

I. Evaluate the following :

$$1. \int x^2 \cdot \log x \, dx$$

$$2. \int x^2 \cdot \sin 3x \, dx$$

$$3. \int x \cdot \tan^{-1} x \, dx$$

$$4. \int x^2 \cdot \tan^{-1} x \, dx$$

$$5. \int x^3 \cdot \tan^{-1} x \, dx$$

$$6. \int (\log x)^2 \, dx$$

$$7. \int \sec^3 x \, dx$$

$$8. \int x \cdot \sin^2 x \, dx$$

$$9. \int x^3 \cdot \log x \, dx$$

$$10. \int e^{2x} \cdot \cos 3x \, dx$$

$$11. \int x \cdot \sin^{-1} x \, dx$$

$$12. \int x^2 \cdot \cos^{-1} x \, dx$$

$$13. \int \frac{\log(\log x)}{x} dx$$

$$14. \int \frac{t \sin^{-1} t}{\sqrt{1-t^2}} dt$$

$$15. \int \cos \sqrt{x} \, dx$$

$$16. \int \sin \theta \cdot \log(\cos \theta) d\theta$$

$$17. \int x \cdot \cos^3 x \, dx$$

$$18. \int \frac{\sin(\log x)^2}{x} \cdot \log x \, dx$$

$$19. \int \frac{\log x}{x} dx$$

$$20. \int x \cdot \sin 2x \cdot \cos 5x \, dx$$

$$21. \int \cos(\sqrt[3]{x}) \, dx$$

II. Integrate the following functions w. r. t. x :

1. $e^{2x} \cdot \sin 3x$
2. $e^{-x} \cdot \cos 2x$
3. $\sin (\log x)$
4. $\sqrt{5x^2 + 3}$
5. $x^2 \cdot \sqrt{a^2 - x^6}$
6. $\sqrt{(x-3)(7-x)}$
7. $\sqrt{4^x(4^x + 4)}$
8. $(x+1)\sqrt{2x^2 + 3}$
9. $x \sqrt{5 - 4x - x^2}$
10. $\sec^2 x \cdot \sqrt{\tan^2 x + \tan x - 7}$
11. $\sqrt{x^2 + 2x + 5}$
12. $\sqrt{2x^2 + 3x + 4}$

III. Integrate the following functions w. r. t. x :

1. $(2 + \cot x - \operatorname{cosec}^2 x) \cdot e^x$
2. $\left(\frac{1 + \sin x}{1 + \cos x} \right) \cdot e^x$
3. $e^x \cdot \left(\frac{1}{x} - \frac{1}{x^2} \right)$
4. $\left(\frac{x}{(x+1)^2} \right) \cdot e^x$
5. $\frac{e^x}{x} [x (\log x)^2 + 2 (\log x)]$
6. $e^{5x} \cdot \left(\frac{5x \cdot \log x + 1}{x} \right)$
7. $e^{\sin^{-1} x} \cdot \left(\frac{x + \sqrt{1-x^2}}{\sqrt{1-x^2}} \right)$
8. $\log (1+x)^{(1+x)}$
9. $\operatorname{cosec} (\log x) [1 - \cot (\log x)]$

3.4 Integration by partial fraction :

If $f(x)$ and $g(x)$ are two polynomials then $\frac{f(x)}{g(x)}$, $g(x) \neq 0$ is called a rational algebraic function.

$\frac{f(x)}{g(x)}$ is called a proper rational function provided degree of $f(x) <$ degree of $g(x)$; otherwise it is called **improper rational function**.

If degree of $f(x) \geq$ degree of $g(x)$ i.e. $\frac{f(x)}{g(x)}$ is an improper rational function then express it as in the form $\text{Quotient} + \frac{\text{Remainder}}{g(x)}$, $g(x) \neq 0$ where $\frac{\text{Remainder}}{g(x)}$ is proper rational function.

Lets see the three different types of the proper rational function $\frac{f(x)}{g(x)}$, $g(x) \neq 0$ where the denominator $g(x)$ is expressed as

- (i) a non-repeated linear factors
- (ii) repeated Linear factors and
- (iii) product of Linear factor and non-repeated quadratic factor.

No.	Rational form	Partial form
(i)	$\frac{px^2 + qx + r}{(x-a)(x-b)(x-c)}$	$\frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)}$
(ii)	$\frac{px^2 + qx + r}{(x-a)^2(x-b)}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-b)}$
(iii)	$\frac{px^2 + qx + r}{(x-a)(x^2 + bx + c)}$	$\frac{A}{(x-a)} + \frac{Bx + C}{x^2 + bx + c}$

Type (i) : $\int \frac{px^2 + qx + r}{(x-a)(x-b)(x-c)} dx$ i.e. denominator is expressed as non-repeated Linear factors.



SOLVED EXAMPLES

1. $\int \frac{3x^2 + 4x - 5}{(x^2 - 1)(x + 2)} dx$

Solution : $I = \int \frac{3x^2 + 4x - 5}{(x-1)(x+1)(x+2)} dx$

Consider, $\frac{3x^2 + 4x - 5}{(x-1)(x+1)(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(x+2)}$

$$= \frac{A(x+1)(x+2) + B(x-1)(x+2) + C(x-1)(x+1)}{(x-1)(x+1)(x+2)}$$

$$\therefore 3x^2 + 4x - 5 = A(x+1)(x+2) + B(x-1)(x+2) + C(x-1)(x+1)$$

at $x = 1$, $3(1)^2 + 4(1) - 5 = A(2)(3) + B(0) + C(0)$

$$2 = 6A \Rightarrow A = \frac{1}{3}$$

at $x = -1$, $3(-1)^2 + 4(-1) - 5 = A(0) + B(-2)(1) + C(0)$

$$-6 = -2B \Rightarrow B = 3$$

at $x = -2$, $3(-2)^2 + 4(-2) - 5 = A(0) + B(0) + C(-3)(-1)$

$$-1 = 3C \Rightarrow C = -\frac{1}{3}$$

Thus, $\frac{3x^2 + 4x - 5}{(x-1)(x+1)(x+2)} = \frac{\left(\frac{1}{3}\right)}{(x-1)} + \frac{3}{(x+1)} + \frac{\left(-\frac{1}{3}\right)}{(x+2)}$

$$\therefore I = \int \left[\frac{\left(\frac{1}{3}\right)}{(x-1)} + \frac{3}{(x+1)} + \frac{\left(-\frac{1}{3}\right)}{(x+2)} \right] dx = \frac{1}{3} \log(x-1) + 3 \log(x+1) - \frac{1}{3} \log(x+2) + c$$

$$= \frac{1}{3} \log \left[\frac{(x-1)(x+1)^9}{(x+2)} \right] + c \quad \therefore \int \frac{3x^2 + 4x - 5}{(x^2 - 1)(x+2)} dx = \frac{1}{3} \log \left[\frac{(x-1)(x+1)^9}{(x+2)} \right] + c$$

2. $\int \frac{2x^2 - 3}{(x^2 - 5)(x^2 + 4)} dx$

Solution : Consider, $\frac{2x^2 - 3}{(x^2 - 5)(x^2 + 4)}$

Let $x^2 = m$

$$\therefore = \frac{2m-3}{(m-5)(m+4)} \dots \text{proper rational function.}$$

$$\text{Now, } \frac{2m-3}{(m-5)(m+4)} = \frac{A}{(m-5)} + \frac{B}{(m+4)} = \frac{A(m+4) + B(m-5)}{(m-5)(m+4)}$$

$$\therefore 2m - 3 = A(m + 4) + B(m - 5)$$

$$\text{at } m = 5, \quad 2(5) - 3 = A(9) + B(0)$$

$$7 = 9A \quad \Rightarrow \quad A = \frac{7}{9}$$

$$\text{at } m = -4, \quad 2(-4) - 3 = A(0) + B(-9)$$

$$-11 = -9B \Rightarrow B = \frac{11}{9}$$

$$\text{Thus, } \frac{2m-3}{(m-5)(m+4)} = \frac{\left(\frac{7}{9}\right)}{(m-5)} + \frac{\left(\frac{11}{9}\right)}{(m+4)} \quad \text{i.e. } \frac{2x^2-3}{(x^2-5)(x^2+4)} = \frac{\left(\frac{7}{9}\right)}{x^2-5} + \frac{\left(\frac{11}{9}\right)}{x^2+4}$$

$$\begin{aligned}\therefore \text{I} &= \int \left[\frac{\left(\frac{7}{9}\right)}{x^2 - 5} + \frac{\left(\frac{11}{9}\right)}{x^2 + 4} \right] dx \\ &= \frac{7}{9} \cdot \int \frac{1}{x^2 - (\sqrt{5})^2} dx + \frac{11}{9} \cdot \int \frac{1}{x^2 + (2)^2} dx \\ &= \frac{7}{9} \cdot \frac{1}{2(\sqrt{5})} \cdot \log \left[\frac{x - \sqrt{5}}{x + \sqrt{5}} \right] + \frac{11}{9} \cdot \frac{1}{2} \cdot \tan^{-1} \left(\frac{x}{2} \right) + c\end{aligned}$$

$$\therefore I = \frac{7}{18(\sqrt{5})} \cdot \log \left[\frac{x - \sqrt{5}}{x + \sqrt{5}} \right] + \frac{11}{18} \cdot \tan^{-1} \left(\frac{x}{2} \right) + c$$

$$\therefore \int \frac{2x^2 - 3}{(x^2 - 5)(x^2 + 4)} dx = \frac{7}{18(\sqrt{5})} \cdot \log \left[\frac{x - \sqrt{5}}{x + \sqrt{5}} \right] + \frac{11}{18} \cdot \tan^{-1} \left(\frac{x}{2} \right) + c$$

3. $\int \frac{1}{(\sin \theta) (3 + 2 \cos \theta)} d\theta$

Solution : $I = \int \frac{1}{(\sin \theta) (3 + 2 \cos \theta)} d\theta = \int \frac{\sin \theta}{(1 - \cos^2 \theta) (3 + 2 \cos \theta)} d\theta$
 $= \int \frac{\sin \theta}{(1 - \cos \theta) (1 + \cos \theta) (3 + 2 \cos \theta)} d\theta$

put $\cos \theta = t \quad \therefore -\sin \theta \cdot d\theta = 1 dt$

$\therefore \sin \theta \cdot d\theta = -1 dt$

Consider, $\frac{-1}{(1-t)(1+t)(3+2t)} = \frac{A}{(1-t)} + \frac{B}{(1+t)} + \frac{C}{(3+2t)}$
 $= \frac{A(1+t)(3+2t) + B(1-t)(3+2t) + C(1-t)(1+t)}{(1-t)(1+t)(3+2t)}$

$\therefore -1 = A(1+t)(3+2t) + B(1-t)(3+2t) + C(1-t)(1+t)$

at $t = 1$, $-1 = A(2)(5) + B(0) + C(0)$

$-1 = 10A \Rightarrow A = -\frac{1}{10}$

at $t = -1$, $-1 = A(0) + B(2)(1) + C(0)$

$-1 = 2B \Rightarrow B = -\frac{1}{2}$

at $t = -\frac{3}{2}$, $-1 = A(0) + B(0) + C\left(+\frac{5}{2}\right)\left(-\frac{1}{2}\right)$

$-1 = -\frac{5}{4}C \Rightarrow C = \frac{4}{5}$

Thus, $\frac{-1}{(1-t)(1+t)(3+2t)} = \frac{\left(-\frac{1}{10}\right)}{(1-t)} + \frac{\left(-\frac{1}{2}\right)}{(1+t)} + \frac{\left(\frac{4}{5}\right)}{(3+2t)}$

$\therefore I = \int \left[\frac{\left(-\frac{1}{10}\right)}{(1-t)} + \frac{\left(-\frac{1}{2}\right)}{(1+t)} + \frac{\left(\frac{4}{5}\right)}{(3+2t)} \right] dt$
 $= -\frac{1}{10} \log(1-t) \cdot \frac{1}{(-1)} - \frac{1}{2} \log(1+t) + \frac{4}{5} \log(3+2t) \cdot \frac{1}{2} + c$
 $= \frac{1}{10} \log(1 - \cos \theta) - \frac{1}{2} \log(1 + \cos \theta) + \frac{4}{5} \log(3 + 2 \cos \theta) + c$
 $= \frac{1}{10} \left(\log \frac{(1 - \cos \theta) (3 + 2 \cos \theta)^4}{(1 + \cos \theta)^5} \right) + c \quad \because \log a^m = m \cdot \log a$

4. $\int \frac{1}{2 \cos x + \sin 2x} dx$

Solution : $I = \int \frac{1}{2 \cos x + \sin 2x} dx = \int \frac{1}{2 \cos x + 2 \sin x \cdot \cos x} dx = \int \frac{1}{2 (\cos x) (1 + \sin x)} dx$

$= \frac{1}{2} \cdot \int \frac{\cos x}{\cos^2 x (1 + \sin x)} dx = \frac{1}{2} \int \frac{\cos x}{(1 - \sin^2 x) (1 + \sin x)} dx$

$$\text{put } \sin x = t \quad \therefore \quad \cos x \, dx = 1 \, dt$$

$$= \frac{1}{2} \cdot \int \frac{1}{(1-t^2)(1+t)} dt = \frac{1}{2} \cdot \int \frac{1}{(1-t)(1+t)(1+t)} dt = \frac{1}{2} \cdot \int \frac{1}{(1-t)(1+t)^2} dt$$

Consider,
$$\frac{1}{(1-t)(1+t)^2} = \frac{A}{(1-t)} + \frac{B}{(1+t)} + \frac{C}{(1+t)^2} = \frac{A(1+t)^2 + B(1-t)(1+t) + C(1-t)}{(1-t)(1+t)^2}$$

$$\therefore 1 = A(1+t)^2 + B(1-t)(1+t) + C(1-t)$$

$$\text{at } t = 1, \quad 1 = A(2)^2 + B(0) + C(0)$$

$$1 = 4A \quad \Rightarrow \quad A = \frac{1}{4}$$

$$\text{at } t = -1, \quad 1 = A(0) + B(0) + C(2)$$

$$1 = 2C \quad \Rightarrow \quad C = \frac{1}{2}$$

$$\text{at } t = 0, \quad 1 = A(1)^2 + B(1)(1) + C(1)$$

$$1 = A + B + C$$

$$1 = \frac{1}{4} + B + \frac{1}{2} \Rightarrow B = \frac{1}{4}$$

$$\text{Thus, } \frac{1}{(1-t)(1+t)^2} = \frac{\left(\frac{1}{4}\right)}{(1-t)} + \frac{\left(\frac{1}{4}\right)}{(1+t)} + \frac{\left(\frac{1}{2}\right)}{(1+t)^2}$$

$$\therefore I = \int \left[\frac{\left(\frac{1}{4}\right)}{(1-t)} + \frac{\left(\frac{1}{4}\right)}{(1+t)} + \frac{\left(\frac{1}{2}\right)}{(1+t)^2} \right] dt = \frac{1}{2} \left[\frac{1}{4} \log(1-t) \cdot \frac{1}{(-1)} + \frac{1}{4} \log(1+t) + \frac{1}{2} \cdot \frac{(-1)}{(1+t)} \right] + c$$

$$= \frac{1}{2} \left[\frac{1}{4} \log(1-t) \cdot \frac{1}{(-1)} + \frac{1}{4} \log(1+t) + \frac{1}{2} \cdot \frac{-1}{1+t} \right] + c$$

$$= \frac{1}{8} \left[-\log(1 - \sin x) + \log(1 + \sin x) - \frac{2}{1 + \sin x} \right] + c = \frac{1}{8} \left[\log \left(\frac{1 + \sin x}{1 - \sin x} \right) - \frac{2}{1 + \sin x} \right] + c$$

$$\therefore \int \frac{1}{2 \cos x + \sin 2x} \cdot dx = \frac{1}{8} \left[\log \left(\frac{1 + \sin x}{1 - \sin x} \right) - \frac{2}{1 + \sin x} \right] + c$$

5. $\int \frac{\tan \theta + \tan^3 \theta}{1 + \tan^3 \theta} d\theta$

Solution : $I = \int \frac{(\tan \theta) (1 + \tan^2 \theta)}{1 + \tan^3 \theta} d\theta = \int \frac{(\tan \theta) (1 + \tan^2 \theta)}{1 + \tan^3 \theta} d\theta = \int \frac{\tan \theta \sec^2 \theta}{1 + \tan^3 \theta} d\theta$

put $\tan \theta = x \quad \therefore \sec^2 \theta d\theta = 1 dx$

$$= \int \frac{x}{1 + x^3} dx = \int \frac{x}{(1 + x)(1 - x + x^2)} dx$$

Consider, $\frac{x}{(1 + x)(1 - x + x^2)} = \frac{A}{1 + x} + \frac{Bx + C}{(1 - x + x^2)}$

$$= \frac{A(1 - x + x^2) + Bx + C(1 + x)}{(1 + x)(1 - x + x^2)}$$

$$\therefore x = A(1 - x + x^2) + (Bx + C)(1 + x) = A - Ax + Ax^2 + Bx + Bx^2 + C + Cx$$

$$0x^2 + 1x + 0 = (A + B)x^2 + (-A + B + C)x + (A + C)$$

comparing the co-efficients of like powers of variables.

$$0 = A + B \quad \dots \text{(I)}$$

$$1 = -A + B + C \quad \dots \text{(II)} \quad \text{and}$$

$$0 = A + C \quad \dots \text{(III)}$$

Solving these equations, we get $A = -\frac{1}{3}$; $B = \frac{1}{3}$ and $C = \frac{1}{3}$

Thus, $\frac{x}{(1 + x)(1 - x + x^2)} = \frac{\left(-\frac{1}{3}\right)}{1 + x} + \frac{\left(\frac{1}{3}x + \frac{1}{3}\right)}{(1 - x + x^2)}$

$$\begin{aligned} \therefore I &= \int \left[\frac{\left(-\frac{1}{3}\right)}{1 + x} + \frac{\left(\frac{1}{3}x + \frac{1}{3}\right)}{(1 - x + x^2)} \right] dx = -\frac{1}{3} \cdot \int \frac{1}{1 + x} dx + \frac{1}{3} \cdot \int \frac{x + 1}{1 - x + x^2} dx \\ &= -\frac{1}{3} \cdot \int \frac{1}{1 + x} dx + \frac{1}{3} \cdot \frac{1}{(2)} \int \frac{2x - 1 + 3}{x^2 - x + 1} dx \quad \because \frac{d}{dx} x^2 - x + 1 = 2x - 1 \\ &= -\frac{1}{3} \cdot \int \frac{1}{1 + x} dx + \frac{1}{3} \cdot \frac{1}{2} \cdot \int \frac{2x - 1 + 3}{x^2 - x + 1} dx \\ &= -\frac{1}{3} \cdot \int \frac{1}{1 + x} dx + \frac{1}{6} \cdot \int \frac{2x - 1}{x^2 - x + 1} dx + \frac{1}{6} \cdot \int \frac{3}{x^2 - x + 1} dx \\ &= I_1 + I_2 + I_3 \quad \dots \text{(IV)} \end{aligned}$$

$$\begin{aligned}\therefore I_1 &= -\frac{1}{3} \int \frac{1}{1+x} dx = -\frac{1}{3} [\log(1+x)] \\ &= -\frac{1}{3} \log(1 + \tan \theta) + c_1 \quad \dots \text{(V)}\end{aligned}$$

$$\begin{aligned}\therefore I_2 &= \frac{1}{6} \int \frac{2x-1}{x^2-x+1} dx = \frac{1}{6} [\log(x^2-x+1)] + c_2 \\ &= \frac{1}{6} \log(\tan^2 \theta - \tan \theta + 1) + c_2 \quad \dots \text{(VI)}\end{aligned}$$

$$\begin{aligned}\therefore I_3 &= \frac{1}{6} \int \frac{3}{x^2-x+1} dx \\ &= \frac{1}{2} \int \frac{1}{x^2-x+\frac{1}{4}-\frac{1}{4}+1} dx \quad \because \left\{ \left(\frac{1}{2} \text{ coefficient of } x \right)^2 = \left(\frac{1}{2}(-1) \right)^2 = \left(-\frac{1}{2} \right)^2 = \frac{1}{4} \right\} \\ &= \frac{1}{2} \int \frac{1}{\left(x - \frac{1}{2} \right)^2 + \left(\frac{\sqrt{3}}{2} \right)^2} dx \\ &= \frac{1}{2} \left[\frac{1}{\left(\frac{\sqrt{3}}{2} \right)} \right] \tan^{-1} \left[\frac{x - \frac{1}{2}}{\left(\frac{\sqrt{3}}{2} \right)} \right] + c \\ &= \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{2x-1}{\sqrt{3}} \right) + c \\ \therefore I_3 &= \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{2 \tan \theta - 1}{\sqrt{3}} \right) + c \quad \dots \text{(VII)}\end{aligned}$$

$$\therefore \int \frac{\tan \theta + \tan^3 \theta}{1 + \tan^3 \theta} d\theta = -\frac{1}{3} \log(1 + \tan \theta) + \frac{1}{6} \log(\tan^2 \theta - \tan \theta + 1) + \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{2 \tan \theta - 1}{\sqrt{3}} \right) + c$$

EXERCISE 3.4

I. Integrate the following w. r. t. x :

1. $\frac{x^2 + 2}{(x-1)(x+2)(x+3)}$

2. $\frac{x^2}{(x^2+1)(x^2-2)(x^2+3)}$

3. $\frac{12x+3}{6x^2+13x-63}$

4. $\frac{2x}{4-3x-x^2}$
5. $\frac{x^2+x-1}{x^2+x-6}$
6. $\frac{6x^3+5x^2-7}{3x^2-2x-1}$
7. $\frac{12x^2-2x-9}{(4x^2-1)(x+3)}$
8. $\frac{1}{x(x^5+1)}$
9. $\frac{2x^2-1}{x^4+9x^2+20}$
10. $\frac{x^2+3}{(x^2-1)(x^2-2)}$
11. $\frac{2x}{(2+x^2)(3+x^2)}$
12. $\frac{2^x}{4^x-3 \cdot 2^x-4}$
13. $\frac{3x-2}{(x+1)^2(x+3)}$
14. $\frac{5x^2+20x+6}{x^3+2x^2+x}$
15. $\frac{1}{x(1+4x^3+3x^6)}$
16. $\frac{1}{x^3-1}$
17. $\frac{(3 \sin x - 2) \cdot \cos x}{5 - 4 \sin x - \cos^2 x}$
18. $\frac{1}{\sin x + \sin 2x}$
19. $\frac{1}{2 \sin x + \sin 2x}$
20. $\frac{1}{\sin 2x + \cos x}$
21. $\frac{1}{\sin x \cdot (3 + 2 \cos x)}$
22. $\frac{5 \cdot e^x}{(e^x+1)(e^{2x}+9)}$
23. $\frac{2 \log x + 3}{x(3 \log x + 2)[(\log x)^2 + 1]}$

3.5 Something Interesting :

Students/ now familiar with the integration by parts.

The result is $\int u v dx = u \int v dx - \int \left(\frac{d}{dx} u \right) (\int v dx) dx$,

u and v are differentiable functions of x and $u \cdot v$ follows L I A T E order.

This result can be extended to the generalisation as -

$$\int u v dx = u v_1 - u' v_2 + u'' v_3 - u''' v_4 + \dots$$

(') dash indicates the derivative.

(₁) subscript indicates the integration.

This result is more useful where the first function (u) is a polynomial, because $\frac{d^n u}{dx^n} = 0$ for some n .

For example : $\int x^2 \cos 3x dx$

$$\begin{aligned} &= x^2 \left(\sin 3x \frac{1}{3} \right) - (2x) \left(-\cos 3x \frac{1}{3} \frac{1}{3} \right) + (2) \left(-\sin 3x \frac{1}{3} \frac{1}{9} \right) - (0) \\ &= \frac{1}{3} x^2 \sin 3x + \frac{2}{9} x \cos 3x - \frac{2}{27} \sin 3x + c \end{aligned}$$

verify this example with usual rule of integration by parts.



Let us Remember

✿ We can always add arbitrary constant c to the integration obtained :

(I) i.e. $\frac{d}{dx} g(x) = f(x) \Rightarrow \int f(x) dx = g(x) + c$
 $f(x)$ is integrand, $g(x)$ is integral of $f(x)$ with respect to x , c is arbitrary constant.

(II) $\int f(ax + b) dx = g(ax + b) \frac{1}{a} + c$

(III) (1) $\int [f(x)]^n f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + c$ (2) $\int \frac{f'(x)}{f(x)} dx = \log(f(x)) + c$

(3) $\int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + c$

(IV) (1) $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

(2) $\int \frac{1}{\sqrt{x}} dx = 2\sqrt{x} + c$

(3) $\int (k) dx = kx + c$

(4) $\int a^x dx = \frac{a^x}{\log a} + c$

(5) $\int e^x dx = e^x + c$

(6) $\int \frac{1}{x} dx = \log(x) + c$

(7) $\int \sin x dx = -\cos x + c$

(8) $\int \cos x dx = \sin x + c$

(9) $\int \tan x dx = \log(\sec x) + c$

(10) $\int \cot x dx = \log(\sin x) + c$

(11) $\int \sec x dx = \log(\sec x + \tan x) + c$ (12) $\int \operatorname{cosec} x dx = \log(\operatorname{cosec} x - \cot x) + c$

$= \log \left[\tan \left(\frac{x}{2} + \frac{\pi}{4} \right) \right] + c$ $= \log \left[\tan \left(\frac{x}{2} \right) \right] + c$

(13) $\int \sec^2 x dx = \tan x + c$

(14) $\int \operatorname{cosec}^2 x dx = -\cot x + c$

(15) $\int \sec x \tan x dx = \sec x + c$

(16) $\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + c$

(17) $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$

(18) $\int \frac{-1}{\sqrt{1-x^2}} dx = \cos^{-1} x + c$

(19) $\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$

(20) $\int \frac{-1}{1+x^2} dx = \cot^{-1} x + c$

(21) $\int \frac{1}{x \sqrt{x^2-1}} dx = \sec^{-1} x + c$

(22) $\int \frac{-1}{x \sqrt{x^2-1}} dx = \operatorname{cosec}^{-1} x + c$

$$\begin{aligned}
 (23) \quad \int \frac{1}{x^2 + a^2} dx &= \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c & (24) \quad \int \frac{1}{x^2 - a^2} dx &= \frac{1}{2a} \log \left(\frac{x-a}{x+a} \right) + c \\
 (25) \quad \int \frac{1}{a^2 - x^2} dx &= \frac{1}{2a} \log \left(\frac{a+x}{a-x} \right) + c & (26) \quad \int \frac{1}{\sqrt{a^2 - x^2}} dx &= \sin^{-1} \left(\frac{x}{a} \right) + c \\
 (27) \quad \int \frac{1}{\sqrt{x^2 - a^2}} dx &= \log |x + \sqrt{x^2 - a^2}| + c & (28) \quad \int \frac{1}{\sqrt{x^2 + a^2}} dx &= \log |x + \sqrt{x^2 + a^2}| + c \\
 (29) \quad \int \frac{1}{x\sqrt{x^2 - a^2}} dx &= \frac{1}{a} \sec^{-1} \left(\frac{x}{a} \right) + c \\
 (30) \quad \int \sqrt{a^2 - x^2} dx &= \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + c \\
 (31) \quad \int \sqrt{a^2 + x^2} dx &= \frac{x}{2} \sqrt{a^2 + x^2} + \frac{a^2}{2} \log (x + \sqrt{x^2 + a^2}) + c \\
 (32) \quad \int \sqrt{x^2 - a^2} dx &= \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log (x + \sqrt{x^2 - a^2}) + c
 \end{aligned}$$

(V) If u and v are differentiable functions of x then $\int u v dx = u \int v dx - \int \left(\frac{d}{dx} u \right) (\int v dx) dx$ where $u v$ follows the L I A T E order.

(VI) $\int e^x [f(x) + f'(x)] dx = e^x f(x) + c$

(VII) For the integration of type $\int \frac{f(x)}{g(x)} dx$, $g(x) \neq 0$ where $\frac{f(x)}{g(x)}$ proper rational function.

(i) non-repeated linear factors

(ii) repeated Linear factors and

(iii) product of Linear factor and non-repeated quadratic factor.

(VIII) $\int \frac{1}{x^2 + a^2} dx$ $\int \frac{1}{x^2 - a^2} dx$ $\int \frac{1}{a^2 - x^2} dx$ $\int \frac{1}{ax^2 + bx + c} dx$ $\int \frac{1}{a \sin^2 x + b \cos^2 x + c} dx$

Method of completing square

Divide Nr and Dr by $\cos^2 x$

$\int \frac{1}{a \sin x + b \cos x + c} dx$

put $\tan \left(\frac{x}{2} \right) = t$

$\int \frac{px + q}{ax^2 + bx + c} dx$

$px + q = A \frac{d}{dx} (ax^2 + bx + c) + B$

(VIII) $\int \frac{1}{\sqrt{x^2 + a^2}} dx$ $\int \frac{1}{\sqrt{x^2 - a^2}} dx$ $\int \frac{1}{\sqrt{a^2 - x^2}} dx$ $\int \frac{1}{\sqrt{ax^2 + bx + c}} dx$ $\int \frac{px + q}{\sqrt{ax^2 + bx + c}} dx$

MISCELLANEOUS EXERCISE 3

(I) Choose the correct option from the given alternatives :

(1) $\int \frac{1+x+\sqrt{x+x^2}}{\sqrt{x}+\sqrt{1+x}} dx =$

- (A) $\frac{1}{2} \sqrt{x+1} + c$ (B) $\frac{2}{3} (x+1)^{\frac{3}{2}} + c$ (C) $\sqrt{x+1} + c$ (D) $2(x+1)^{\frac{3}{2}} + c$

(2) $\int \frac{1}{x+x^5} dx = f(x) + c$, then $\int \frac{x^4}{x+x^5} dx =$

- (A) $\log x - f(x) + c$ (B) $f(x) + \log x + c$ (C) $f(x) - \log x + c$ (D) $\frac{1}{5} x^5 f(x) + c$

(3) $\int \frac{\log(3x)}{x \log(9x)} dx =$

- (A) $\log(3x) - \log(9x) + c$ (B) $\log(x) - (\log 3) \cdot \log(\log 9x) + c$
(C) $\log 9 - (\log x) \cdot \log(\log 3x) + c$ (D) $\log(x) + (\log 3) \cdot \log(\log 9x) + c$

(4) $\int \frac{\sin^m x}{\cos^{m+2} x} dx =$

- (A) $\frac{\tan^{m+1} x}{m+1} + c$ (B) $(m+2) \tan^{m+1} x + c$ (C) $\frac{\tan^m x}{m} + c$ (D) $(m+1) \tan^{m+1} x + c$

(5) $\int \tan(\sin^{-1} x) dx =$

- (A) $(1-x^2)^{-\frac{1}{2}} + c$ (B) $(1-x^2)^{\frac{1}{2}} + c$ (C) $\frac{\tan^m x}{\sqrt{1-x^2}} + c$ (D) $-\sqrt{1-x^2} + c$

(6) $\int \frac{x - \sin x}{1 - \cos x} dx =$

- (A) $x \cot\left(\frac{x}{2}\right) + c$ (B) $-x \cot\left(\frac{x}{2}\right) + c$ (C) $\cot\left(\frac{x}{2}\right) + c$ (D) $x \tan\left(\frac{x}{2}\right) + c$

(7) If $f(x) = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$, $g(x) = e^{\sin^{-1} x}$, then $\int f(x) \cdot g(x) dx =$

- (A) $e^{\sin^{-1} x} (\sin^{-1} x - 1) + c$ (B) $e^{\sin^{-1} x} (1 - \sin^{-1} x) + c$
(C) $e^{\sin^{-1} x} (\sin^{-1} x + 1) + c$ (D) $e^{\sin^{-1} x} (\sin^{-1} x - 1) + c$

(8) If $\int \tan^3 x \sec^3 x dx = \left(\frac{1}{m}\right) \sec^m x - \left(\frac{1}{n}\right) \sec^n x + c$, then $(m, n) =$

- (A) (5, 3) (B) (3, 5) (C) $\left(\frac{1}{5}, \frac{1}{3}\right)$ (D) (4, 4)

- (9) $\int \frac{1}{\cos x - \cos^2 x} dx =$
- (A) $\log (\operatorname{cosec} x - \cot x) + \tan \left(\frac{x}{2} \right) + c$ (B) $\sin 2x - \cos x + c$
- (C) $\log (\sec x + \tan x) - \cot \left(\frac{x}{2} \right) + c$ (D) $\cos 2x - \sin x + c$
- (10) $\int \frac{\sqrt{\cot x}}{\sin x \cos x} dx =$
- (A) $2 \sqrt{\cot x} + c$ (B) $-2 \sqrt{\cot x} + c$ (C) $\frac{1}{2} \sqrt{\cot x} + c$ (D) $\sqrt{\cot x} + c$
- (11) $\int \frac{e^x (x-1)}{x^2} dx =$
- (A) $\frac{e^x}{x} + c$ (B) $\frac{e^x}{x^2} + c$ (C) $\left(x - \frac{1}{x} \right) e^x + c$ (D) $xe^{-x} + c$
- (12) $\int \sin (\log x) dx =$
- (A) $\frac{x}{2} [\sin (\log x) - \cos (\log x)] + c$ (B) $\frac{x}{2} [\sin (\log x) + \cos (\log x)] + c$
- (C) $\frac{x}{2} [\cos (\log x) - \sin (\log x)] + c$ (D) $\frac{x}{4} [\cos (\log x) - \sin (\log x)] + c$
- (13) $\int x^x (1 + \log x) dx =$
- (A) $\frac{1}{2} (1 + \log x)^2 + c$ (B) $x^{2x} + c$ (C) $x^x \log x + c$ (D) $x^x + c$
- (14) $\int \cos^{-\frac{3}{7}} x \sin^{-\frac{11}{7}} x dx =$
- (A) $\log \left(\sin^{-\frac{4}{7}} x \right) + c$ (B) $\frac{4}{7} \tan^{\frac{4}{7}} x + c$
- (C) $-\frac{7}{4} \tan^{-\frac{4}{7}} x + c$ (D) $\log \left(\cos^{\frac{3}{7}} x \right) + c$
- (15) $2 \int \frac{\cos^2 x - \sin^2 x}{\cos^2 x + \sin^2 x} dx =$
- (A) $\sin 2x + c$ (B) $\cos 2x + c$ (C) $\tan 2x + c$ (D) $2 \sin 2x + c$
- (16) $\int \frac{dx}{\cos x \sqrt{\sin^2 x - \cos^2 x}} dx =$
- (A) $\log \left(\tan x - \sqrt{\tan^2 x - 1} \right) + c$ (B) $\sin^{-1} (\tan x) + c$
- (C) $1 + \sin^{-1} (\cot x) + c$ (D) $\log \left(\tan x + \sqrt{\tan^2 x - 1} \right) + c$

