

4. DEFINITE INTEGRATION



Let us Study

- Definite integral as limit of sum.
- Fundamental theorem of integral calculus.
- Methods of evaluation and properties of definite integral.

4.1 Definite integral as limit of sum :

In the last chapter, we studied various methods of finding the primitives or indefinite integrals of given function. We shall now interpret the definite integrals denoted by $\int_a^b f(x) dx$, read as the integral from a to b of the function $f(x)$ with respect to x . Here $a < b$, are real numbers and $f(x)$ is defined on $[a, b]$. At present, we assume that $f(x) \geq 0$ on $[a, b]$ and $f(x)$ is continuous.

$\int_a^b f(x) dx$ is defined as the area of the region bounded by $y = f(x)$, X-axis and the ordinates $x = a$ and $x = b$. If $g(x)$ is the primitive of $f(x)$ then the area is $g(b) - g(a)$.

The reason of the above definition will be clear from the figure 4.1. and the discussion that follows here. We are using the mean value theorem learnt earlier. Divide the interval $[a, b]$ into a equal parts by

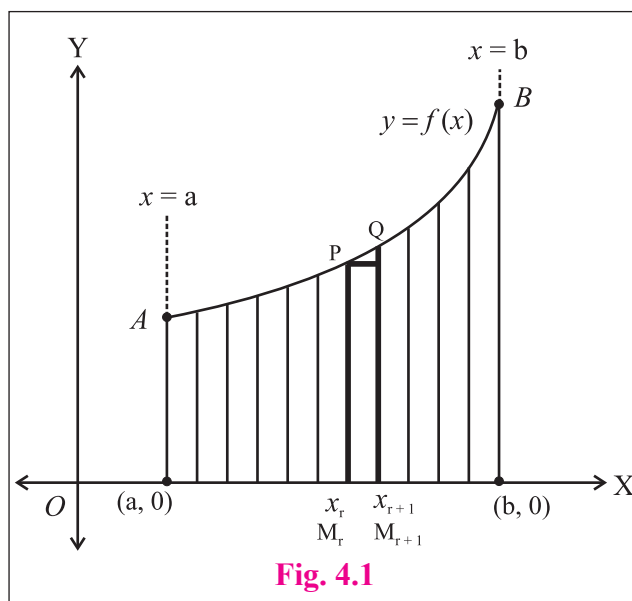


Fig. 4.1

$$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b.$$

Draw the curve $y = f(x)$ in $[a, b]$ and divide the interval $[a, b]$ into n equal parts by

$$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b.$$

Divide the region whose area is measured into their strips as above.

Note that, the area of each strip can be approximated by the area of a rectangle $M_r M_{r+1} QP$ as shown in the figure 4.1, which is $(x_{r+1} - x_r) \times f(T)$ where T is a point on the curve $y = f(x)$ between P and Q .

The mean value theorem states that if $g(x)$ is the primitive of $f(x)$,

$$g(x_{r+1}) - g(x_r) = (x_{r+1} - x_r) \cdot f(t_r) \quad \text{where } x_r < t_r < x_{r+1}.$$

Now we can replace $f(T)$ by $f(t_r)$ given here and express the approximation of the area of the shaded region as $\sum_{r=0}^{n-1} (x_{r+1} - x_r) \cdot f(t_r)$ where $x_r < t_r < x_{r+1}$.

Now we can replace $f(T)$ by $f(t_r)$ given here and express the approximation of the area of the shaded region as

$$\sum_{r=0}^{n-1} (x_{r+1} - x_r) \cdot f(t_r) = \sum_{r=0}^{n-1} g(x_{r+1}) - g(x_r) = g(b) - g(a)$$

Thus taking limit as $n \rightarrow \infty$

$$\begin{aligned} g(b) - g(a) &= \lim_{n \rightarrow \infty} \sum (x_{r+1} - x_r) \cdot f(t_r) \\ &= \lim_{n \rightarrow \infty} S_n \\ &= \int_a^b f(x) dx \end{aligned}$$

The word 'to integrate' means 'to find the sum of'. The technique of integration is very useful in finding plane areas, length of arcs, volume of solid revolution etc...



SOLVED EXAMPLES

Ex. 1 : $\int_1^2 (2x + 5) dx$

Solution : Given, $\int_1^2 (2x + 5) dx = \int_a^b f(x) dx$

$$f(x) = 2x + 5 \quad a = 1 ; b = 2$$

$$\begin{aligned} \Rightarrow f(a + rh) &= f(1 + rh) & \text{and} & \quad h = \frac{b - a}{n} \\ &= 2(1 + rh) + 5 \\ &= 2 + 2rh + 5 & h &= \frac{2 - 1}{n} \\ &= 7 + 2rh & \therefore & \quad nh = 1 \end{aligned}$$

We know $\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{r=1}^n h \cdot f(a + rh)$

$$\begin{aligned}
\therefore \int_1^2 (2x + 5) dx &= \lim_{n \rightarrow \infty} \sum_{r=1}^n h \cdot (7 + 2rh) \\
&= \lim_{n \rightarrow \infty} \sum_{r=1}^n (7h + 2rh^2) \\
&= \lim_{n \rightarrow \infty} \left(7h \sum_{r=1}^n 1 + 2h^2 \sum_{r=1}^n r \right) \\
&= \lim_{n \rightarrow \infty} \left[7h \cdot (n) + 2h^2 \left(\frac{n(n+1)}{2} \right) \right] \\
&= \lim_{n \rightarrow \infty} \left[7nh + h^2 n^2 \left(1 + \frac{1}{n} \right) \right] \\
&= \lim_{n \rightarrow \infty} \left[7(1) + (1)^2 \left(1 + \frac{1}{n} \right) \right] \\
&= 7 + 1(1 + 0) = 8
\end{aligned}$$

Ex. 2 : $\int_2^3 7^x dx$

Solution : Given, $\int_2^3 7^x dx = \int_a^b f(x) dx$

$$f(x) = 7^x \quad a = 2 ; b = 3$$

$$\Rightarrow f(a + rh) = f(1 + rh) \quad \text{and} \quad h = \frac{b-a}{n}$$

$$= 7^{2+rh} \quad h = \frac{3-2}{n} \quad \therefore nh = 1$$

$$= 7^2 \cdot 7^{rh}$$

We know $\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{r=1}^n h \cdot f(a + rh)$

$$\begin{aligned}
\therefore \int_2^3 7^x dx &= \lim_{n \rightarrow \infty} \sum_{r=1}^n h \cdot (7^2 \cdot 7^{r \cdot h}) \\
&= \lim_{n \rightarrow \infty} 7^2 \cdot \sum_{r=1}^n h \cdot 7^{r \cdot h} \\
&= \lim_{n \rightarrow \infty} 7^2 \cdot h \cdot [7^h + 7^{2h} + 7^{3h} + 7^{4h} + \dots + 7^{nh}] \\
&= \lim_{n \rightarrow \infty} 7^2 \cdot h \cdot \left(\frac{7^h [(7^h)^n - 1]}{7^h - 1} \right) = \lim_{n \rightarrow \infty} 7^2 \cdot \left(\frac{7^h (7^{nh} - 1)}{7^h - 1} \right) \\
&= \lim_{n \rightarrow \infty} 7^2 \cdot \left(\frac{7^h (7^{(1)} - 1)}{7^h - 1} \right) \\
&= \frac{7^2 \cdot 7^0 \cdot (7 - 1)}{\log 7} = \frac{(49)(1)(6)}{\log 7} = \frac{294}{\log 7}
\end{aligned}$$

Ex. 3 : $\int_0^4 (x - x^2) \, dx$

Solution : $\int_0^4 (x - x^2) \, dx = \int_a^b f(x) \, dx$

$$f(x) = x - x^2 \quad a = 0 ; b = 4$$

$$\begin{aligned} \Rightarrow f(a + rh) &= f(0 + rh) & \text{and} & \quad h = \frac{b-a}{n} \\ &= f(rh) & & \quad h = \frac{4-0}{n} \\ &= (rh) - (rh)^2 & & \\ &= rh - r^2h^2 & \therefore & \quad nh = 4 \end{aligned}$$

We know $\int_a^b f(x) \, dx = \lim_{n \rightarrow \infty} \sum_{r=1}^n h \cdot [f(a + rh)]$

$$\begin{aligned} \therefore \int_0^4 (x - x^2) \, dx &= \lim_{n \rightarrow \infty} \sum_{r=1}^n h \cdot (rh - r^2h^2) \\ &= \lim_{n \rightarrow \infty} \sum_{r=1}^n (rh^2 - r^2h^3) \\ &= \lim_{n \rightarrow \infty} \left(h^2 \cdot \sum_{r=1}^n r - h^3 \cdot \sum_{r=1}^n r^2 \right) \\ &= \lim_{n \rightarrow \infty} \left[h^2 \left(\frac{n(n+1)}{2} \right) - h^3 \left(\frac{n(n+1)(2n+1)}{6} \right) \right] \\ &= \lim_{n \rightarrow \infty} \left[\frac{h^2 \cdot n \cdot n \left(1 + \frac{1}{n} \right)}{2} - \frac{h^3 \cdot n \cdot n \left(1 + \frac{1}{n} \right) n \left(2 + \frac{1}{n} \right)}{6} \right] \\ &= \lim_{n \rightarrow \infty} \left[\frac{(nh)^2 \left(1 + \frac{1}{n} \right)}{2} - \frac{(nh)^3 \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right)}{6} \right] \\ &= \lim_{n \rightarrow \infty} \left[\frac{(4)^2 \left(1 + \frac{1}{n} \right)}{2} - \frac{(4)^3 \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right)}{6} \right] \\ &= \frac{(4)^2 \cdot (1+0)}{2} - \frac{(4)^3 (1+0) (2+0)}{6} \\ &= 8 - \frac{(64)(2)}{6} \\ &= -\frac{40}{3} \end{aligned}$$

Ex. 4 : $\int_0^{\pi/2} \sin x \, dx$

Solution : $\int_0^{\pi/2} \sin x \, dx = \int_0^{\pi/2} f(x) \, dx$

$$f(x) = \sin x \quad a = 0 ; b = \frac{\pi}{2}$$

$$\begin{aligned} \Rightarrow f(a + rh) &= \sin(a + rh) \\ &= \sin(0 + rh) \quad \text{and} \quad h = \frac{b-a}{n} = \frac{\frac{\pi}{2} - 0}{n} \\ &= \sin rh \quad \therefore \quad nh = \frac{\pi}{2} \end{aligned}$$

We know $\int_a^b f(x) \, dx = \lim_{n \rightarrow \infty} \sum_{r=1}^n h \cdot [f(a + rh)]$

$$\begin{aligned} \therefore \int_0^{\pi/2} \sin x \, dx &= \lim_{n \rightarrow \infty} \sum_{r=1}^n h \cdot \sin rh \\ &= \lim_{n \rightarrow \infty} h \cdot \sum_{r=1}^n \sin rh \\ &= \lim_{n \rightarrow \infty} h \cdot [\sin h + \sin 2h + \sin 3h + \dots + \sin nh] \quad \dots (I) \end{aligned}$$

Consider,

$$\sum_{r=1}^n \sin rh = \sin h + \sin 2h + \sin 3h + \dots + \sin nh$$

$$= 2 \sin \frac{h}{2} \cdot \sin h + 2 \sin \frac{h}{2} \cdot \sin 2h + 2 \sin \frac{h}{2} \cdot \sin 3h + \dots + 2 \sin \frac{h}{2} \cdot \sin nh$$

$$\therefore 2 \sin A \cdot \sin B = \cos(A - B) - \cos(A + B)$$

$$\begin{aligned} 2 \sin \frac{h}{2} \cdot \sum_{r=1}^n \sin rh &= \left[\left(\cos \frac{h}{2} - \cos \frac{3h}{2} \right) + \left(\cos \frac{3h}{2} - \cos \frac{5h}{2} \right) + \left(\cos \frac{5h}{2} - \cos \frac{7h}{2} \right) + \dots \right. \\ &\quad \left. + \dots + \left(\cos \left(\frac{2n-1}{2} h \right) - \cos \left(\frac{2n+1}{2} h \right) \right) \right] \\ &= \left[\cos \frac{h}{2} - \cos \left(\frac{2n+1}{2} h \right) \right] \\ &= \left[\cos \frac{h}{2} - \cos \left(\frac{2nh}{2} + \frac{h}{2} \right) \right] \\ &= \left[\cos \frac{h}{2} - \cos \left(\frac{\pi}{2} + \frac{h}{2} \right) \right] \quad \because \quad nh = \frac{\pi}{2} \\ &= \left(\cos \frac{h}{2} + \sin \frac{h}{2} \right) \end{aligned}$$

$$\therefore \sum_{r=1}^n \sin rh = \frac{\cos \frac{h}{2} + \sin \frac{h}{2}}{2 \sin \frac{h}{2}}$$

Now from I,

$$\int_0^{\pi/2} \sin x \cdot dx = \lim_{n \rightarrow \infty} \sum_{r=1}^n h \cdot \sin rh$$

$$= \lim_{n \rightarrow \infty} h \cdot \left[\frac{\cos \frac{h}{2} + \sin \frac{h}{2}}{2 \sin \frac{h}{2}} \right]$$

$$\therefore nh = \frac{\pi}{4} \text{ as } n \rightarrow \infty \Rightarrow h \rightarrow 0 \left(\frac{1}{n} \rightarrow 0 \right)$$

$$= \lim_{\substack{n \rightarrow \infty \\ h \rightarrow 0}} \left[\frac{\cos \frac{h}{2} + \sin \frac{h}{2}}{\frac{2 \cdot \sin \frac{h}{2}}{h}} \right]$$

$$= \frac{\cos 0 + \sin 0}{\left(\frac{1}{2} \right)}$$

$$= \frac{1 + 0}{2 \cdot \frac{1}{2}} = 1$$

$$\therefore \int_0^{\pi/2} \sin x \, dx = 1$$

EXERCISE 4.1

I. Evaluate the following integrals as limit of sum.

$$(1) \int_1^3 (3x - 4) \, dx$$

$$(2) \int_0^4 x^2 \, dx$$

$$(3) \int_0^2 e^x \, dx$$

$$(4) \int_0^2 (3x^2 - 1) \, dx$$

$$(5) \int_1^3 x^3 \, dx$$

4.2 Fundamental theorem of integral calculus :

Let f be the continuous function defined on $[a, b]$ and if $\int f(x) dx = g(x) + c$

$$\begin{aligned}\text{then } \int_a^b f(x) dx &= [g(x) + c]_a^b \\ &= [(g(b) + c) - (g(a) + c)] \\ &= g(b) + c - g(a) - c \\ &= g(b) - g(a)\end{aligned}$$

$$\text{Thus } \int_a^b f(x) dx = g(b) - g(a)$$

$$\begin{aligned}\text{Ex. : } \int_2^5 (x^2 - x) dx &= \left[\left(\frac{x^3}{3} - \frac{x^2}{2} \right) \right]_2^5 \\ &= \left[\left(\frac{5^3}{3} - \frac{5^2}{2} \right) - \left(\frac{2^3}{3} - \frac{2^2}{2} \right) \right] \\ &= \frac{125}{3} - \frac{25}{2} - \frac{8}{3} + \frac{4}{2} \\ &= \frac{117}{3} - \frac{21}{2} = \frac{234 - 83}{6} \\ \therefore \int_2^5 (x^2 - x) dx &= \frac{151}{3}\end{aligned}$$

In $\int_a^b f(x) dx$ a is called as a lower limit and b is called as an upper limit.

Now let us discuss some fundamental properties of definite integration.

These properties are very useful in evaluation of the definite integral.

4.2.1

Property I : $\int_a^a f(x) dx = 0$

$$\text{Let } \int f(x) dx = g(x) + c$$

$$\begin{aligned}\therefore \int_a^a f(x) dx &= [g(x) + c]_a^a \\ &= [(g(a) + c) - (g(a) + c)] \\ &= 0\end{aligned}$$

Property II : $\int_a^b f(x) dx = - \int_b^a f(x) dx$

$$\text{Let } \int f(x) dx = g(x) + c$$

$$\begin{aligned}\therefore \int_a^b f(x) dx &= [g(x) + c]_a^b \\ &= [(g(b) + c) - (g(a) + c)] \\ &= g(b) - g(a) \\ &= -[g(a) - g(b)] \\ &= - \int_b^a f(x) dx\end{aligned}$$

$$\text{Thus } \int_a^b f(x) dx = - \int_b^a f(x) dx$$

$$\begin{aligned}\text{Ex. } \int_1^3 x dx &= \left[\frac{x^2}{2} \right]_1^3 \\ &= \frac{3^2}{2} - \frac{1^2}{2} = \frac{9}{2} - \frac{1}{2} = 4\end{aligned}$$

$$\begin{aligned}\text{Ex. } \int_3^1 x dx &= \left[\frac{x^2}{2} \right]_3^1 \\ &= \frac{1^2}{2} - \frac{3^2}{2} = \frac{1}{2} - \frac{9}{2} = -4\end{aligned}$$

Property III : $\int_a^b f(x) dx = \int_a^b f(t) dt$

Let $\int f(x) dx = g(x) + c$

L.H.S. : $\int_a^b f(x) dx = [g(x) + c]_a^b$
 $= [(g(b) + c) - (g(a) + c)]$
 $= g(b) - g(a) \dots\dots (i)$

R.H.S. : $\int_a^b f(t) dt = [g(t) + c]_a^b$
 $= [(g(b) + c) - (g(a) + c)]$
 $= g(b) - g(a) \dots\dots (ii)$

from (i) and (ii)

$$\int_a^b f(x) dx = \int_a^b f(t) dt$$

i.e. definite integration is independent of the variable.

Ex. $\int_{\pi/6}^{\pi/3} \cos x \, dx = \left[\sin x \right]_{\pi/6}^{\pi/3}$
 $= \sin \frac{\pi}{3} - \sin \frac{\pi}{6}$
 $= \frac{\sqrt{3}}{2} - \frac{1}{2}$
 $= \frac{\sqrt{3} - 1}{2}$

Ex. $\int_{\pi/6}^{\pi/3} \cos t \, dt = \left[\sin t \right]_{\pi/6}^{\pi/3}$
 $= \sin \frac{\pi}{3} - \sin \frac{\pi}{6}$
 $= \frac{\sqrt{3}}{2} - \frac{1}{2}$
 $= \frac{\sqrt{3} - 1}{2}$

Property IV : $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$ where $a < c < b$ i.e. $c \in [a, b]$

Let $\int f(x) dx = g(x) + c$

Consider R.H.S. : $\int_a^c f(x) dx + \int_c^b f(x) dx$
 $= [g(x) + c]_a^c + [g(x) + c]_c^b$
 $= [(g(c) + c) - (g(a) + c)] + [(g(b) + c) - (g(c) + c)]$
 $= g(c) + c - g(a) - c + g(b) + c - g(c) - c$
 $= g(b) - g(a)$
 $= [g(x) + c]_a^b$
 $= \int_a^b f(x) dx : \text{L.H.S.}$

Thus $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$ where $a < c < b$

Ex. : $\int_{-1}^5 (2x + 3) dx = \int_{-1}^3 (2x + 3) dx + \int_3^5 (2x + 3) dx$

L.H.S. : $\int_{-1}^5 (2x + 3) dx$

$$= \left[2 \frac{x^2}{2} + 3x \right]_{-1}^5$$

$$= \left[x^2 + 3x \right]_{-1}^5$$

$$= [(5)^2 + 3(5)] - [(-1)^2 + 3(-1)]$$

$$= (25 + 15) - (1 - 3)$$

$$= 40 + 2 = 42$$

R.H.S. : $\int_{-1}^3 (2x + 3) dx + \int_3^5 (2x + 3) dx$

$$= \left[x^2 + 3x \right]_{-1}^3 + \left[x^2 + 3x \right]_3^5$$

$$= [(3)^2 + 3(3)] - [(-1)^2 + 3(-1)] + [(5)^2 + 3(5)] - [(3)^2 + 3(3)]$$

$$= [(9 + 9) - (1 - 3)] + [(25 + 15) - (9 + 9)]$$

$$= 18 + 2 + 40 - 18$$

$$= 42$$

Property V : $\int_a^b f(x) dx = \int_a^b f(a + b - x) dx$

Let $\int f(x) dx = g(x) + c$

Consider R.H.S. : $\int_a^b f(a + b - x) dx$

put $a + b - x = t$ i.e. $x = a + b - t$

$\therefore -dx = dt \Rightarrow dx = -dt$

As $x \rightarrow a \Rightarrow t \rightarrow b$ and $x \rightarrow b \Rightarrow t \rightarrow a$

therefore $= \int_b^a f(t) (-dt)$

$$= - \int_b^a f(t) dt$$

$$= \int_a^b f(t) dt \dots \left(\because \int_a^b f(x) dx = \int_a^b f(x) dx \right)$$

$$= \int_a^b f(x) dx \dots \text{as definite integration is independent of the variable.}$$

$=$ L. H. S.

Thus $\int_a^b f(x) dx = \int_a^b f(a + b - x) dx$

Ex. :

$$\int_{\pi/6}^{\pi/3} \sin^2 x dx$$

$I = \int_{\pi/6}^{\pi/3} \sin^2 x dx \dots (i)$

$$= \int_{\pi/6}^{\pi/3} \sin^2 \left(\frac{\pi}{6} + \frac{\pi}{3} - x \right) dx$$

$$= \int_{\pi/6}^{\pi/3} \sin^2 \left(\frac{\pi}{2} - x \right) dx$$

$I = \int_{\pi/6}^{\pi/3} \cos^2 x dx \dots (ii)$

adding (i) and (ii)

$$2I = \int_{\pi/6}^{\pi/3} \sin^2 x dx + \int_{\pi/6}^{\pi/3} \cos^2 x dx$$

$$2I = \int_{\pi/6}^{\pi/3} (\sin^2 x + \cos^2 x) dx$$

$$2I = \int_{\pi/6}^{\pi/3} 1 dx = \left[x \right]_{\pi/6}^{\pi/3}$$

$$2I = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6} \therefore I = \frac{\pi}{12}$$

$$\int_{\pi/6}^{\pi/3} \sin^2 x dx = \frac{\pi}{12}$$

Property VI : $\int_0^a f(x) dx = \int_0^a f(a-x) dx$

$$\text{Let } \int f(x) dx = g(x) + c$$

$$\text{Consider R.H.S. : } \int_0^a f(a-x) dx$$

$$\text{put } a-x=t \quad \text{i.e.} \quad x=a-t$$

$$\therefore -dx = dt \Rightarrow dx = -dt$$

As x varies from 0 to a , t varies from a to 0

$$\begin{aligned} \text{therefore I} &= \int_a^0 f(t) (-dt) \\ &= -\int_a^0 f(t) dt \\ &= \int_0^a f(t) dt \dots \left(\int_a^b f(x) dx = -\int_b^a f(x) dx \right) \\ &= \int_0^a f(x) dx \dots \text{as definite integration is independent of the variable.} \\ &= \text{L. H. S.} \end{aligned}$$

Thus

$$\int_0^a f(x) dx = \int_0^a f(a-x) dx$$

$$\text{Ex. : } \int_0^{\pi/4} \log(1 + \tan x) dx$$

$$\text{Let } \int_0^{\pi/4} \log(1 + \tan x) dx \dots (i)$$

$$\begin{aligned} I &= \int_0^{\pi/4} \log \left[1 + \tan \left(\frac{\pi}{4} - x \right) \right] dx \\ &= \int_0^{\pi/4} \log \left[1 + \frac{\tan \frac{\pi}{4} - \tan x}{1 + \tan \frac{\pi}{4} \cdot \tan x} \right] dx \\ &= \int_0^{\pi/4} \log \left[1 + \frac{1 - \tan x}{1 + \tan x} \right] dx \\ &= \int_0^{\pi/4} \log \left[\frac{1 + \tan x + 1 - \tan x}{1 + \tan x} \right] dx \\ &= \int_0^{\pi/4} \log \left[\frac{2}{1 + \tan x} \right] dx \\ &= \int_0^{\pi/4} [\log 2 - \log(1 + \tan x)] dx \\ &= \int_0^{\pi/4} (\log 2) dx - \int_0^{\pi/4} \log(1 + \tan x) dx \end{aligned}$$

$$I = (\log 2) \int_0^{\pi/4} 1 dx - I \dots \text{by eq. (i)}$$

$$I + I = (\log 2) \left[x \right]_0^{\pi/4}$$

$$2I = (\log 2) \left[\frac{\pi}{4} - 0 \right]$$

$$\therefore I = \frac{\pi}{8} (\log 2)$$

Thus

$$\int_0^{\pi/4} \log(1 + \tan x) dx = \frac{\pi}{8} (\log 2)$$

Property VII :

$$\int_0^{2a} f(x) dx = \int_0^a f(x) dx + \int_0^a f(2a-x) dx$$

$$\begin{aligned} \text{R.H.S. : } & \int_0^a f(x) dx + \int_0^a f(2a-x) dx \\ &= I_1 + I_2 \quad \dots (i) \end{aligned}$$

$$\text{Consider } I_2 = \int_0^a f(2a-x) dx$$

$$\text{put } 2a-x=t \quad \text{i.e.} \quad x=2a-t$$

$$\therefore -1 dx = 1 dt \Rightarrow dx = -dt$$

As x varies from 0 to $2a$, t varies from $2a$ to 0

$$\begin{aligned} I &= \int_{2a}^a f(t) (-dt) \\ &= -\int_{2a}^a f(t) dt \\ &= \int_0^{2a} f(t) dt \dots \left(\int_a^b f(x) dx = -\int_b^a f(x) dx \right) \\ &= \int_0^{2a} f(x) dx \dots \left(\int_a^b f(x) dx = \int_a^b f(t) dt \right) \end{aligned}$$

$$\therefore \int_0^a f(x) dx = \int_0^{2a} f(x) dx$$

from eq. (i)

$$\begin{aligned} \int_0^a f(x) dx + \int_0^a f(2a-x) dx &= \int_0^a f(x) dx + \int_0^{2a} f(x) dx \\ &= \int_0^{2a} f(x) dx : \text{L.H.S} \end{aligned}$$

Thus,

$$\int_0^{2a} f(x) dx = \int_0^a f(x) dx + \int_0^a f(2a-x) dx$$

Property VIII :

$$\int_{-a}^a f(x) dx = 2 \cdot \int_0^a f(x) dx, \text{ if } f(x) \text{ even function}$$

$$= 0, \text{ if } f(x) \text{ is odd function}$$

$f(x)$ even function if $f(-x) = f(x)$

and $f(x)$ odd function if $f(-x) = -f(x)$

$$\int_{-a}^a f(x) dx = \int_{-a}^0 f(x) dx + \int_0^a f(x) dx \quad \dots (i)$$

$$\text{Consider } \int_{-a}^0 f(x) dx$$

$$\text{put } x = -t \quad \therefore dx = -dt$$

As x varies from $-a$ to 0, t varies from a to 0

$$\begin{aligned} I &= \int_a^0 f(-t) (-dt) = -\int_a^0 f(-t) dt \\ &= \int_0^a f(-t) dt \dots \left(\int_a^b f(x) dx = -\int_b^a f(x) dx \right) \\ &= \int_0^a f(-x) dx \dots \left(\int_a^b f(x) dx = \int_a^b f(t) dt \right) \end{aligned}$$

Equation (i) becomes

$$\begin{aligned} \int_{-a}^a f(x) dx &= \int_0^a f(-x) dx + \int_0^a f(x) dx \\ &= \int_0^a [f(-x) + f(x)] dx \end{aligned}$$

If $f(x)$ is odd function then $f(-x) = -f(x)$, hence

$$\int_{-a}^a f(x) dx = 0$$

If $f(x)$ is even function then $f(-x) = f(x)$, hence

$$\int_{-a}^a f(x) dx = 2 \cdot \int_0^a f(x) dx$$

Hence :

$$\begin{aligned} \int_{-a}^a f(x) dx &= 2 \cdot \int_0^a f(x) dx, \text{ if } f(x) \text{ even function} \\ &= 0, \text{ if } f(x) \text{ is odd function} \end{aligned}$$

Ex. :

1. $\int_{-\pi/4}^{\pi/4} x^3 \sin^4 x \, dx$

Let $f(x) = x^3 \sin^4 x$

$$f(-x) = (-x)^3 [\sin(-x)]^4 = -x^3 [-\sin x]^4 = -x^3 \sin^4 x \\ = -f(x)$$

$f(x)$ is odd function.

$\therefore \int_{-\pi/4}^{\pi/4} x^3 \sin^4 x \, dx = 0$

2. $\int_{-1}^1 \frac{x^2}{1+x^2} \, dx$

Let $f(x) = \frac{x^2}{1+x^2}$

$$f(-x) = \frac{(-x)^2}{1+(-x)^2} \\ = \frac{x^2}{1+x^2}$$

$$= f(x)$$

$f(x)$ is even function.

$$\begin{aligned} \int_{-1}^1 \frac{x^2}{1+x^2} \, dx &= 2 \int_0^1 \frac{x^2}{1+x^2} \, dx \\ &= 2 \int_0^1 \frac{1+x^2-1}{1+x^2} \, dx \\ &= 2 \int_0^1 \left[1 - \frac{1}{1+x^2} \right] \, dx \\ &= 2 \left[x - \tan^{-1} x \right]_0^1 \\ &= 2 \{ (1 - \tan^{-1} 1) - (0 - \tan^{-1} 0) \} \\ &= 2 \left\{ 1 - \frac{\pi}{4} - 0 \right\} \\ &= 2 \left(1 - \frac{\pi}{4} \right) = \left(\frac{4-\pi}{2} \right) \end{aligned}$$

$$\therefore \int_{-1}^1 \frac{x^2}{1+x^2} \, dx = \frac{4-\pi}{2}$$



SOLVED EXAMPLES

Ex. 1 : $\int_1^3 \frac{1}{\sqrt{2+x} + \sqrt{x}} \, dx$

Solution :
$$\begin{aligned} &= \int_1^3 \left(\frac{1}{\sqrt{2+x} + \sqrt{x}} \right) \left(\frac{\sqrt{2+x} - \sqrt{x}}{\sqrt{2+x} - \sqrt{x}} \right) dx \\ &= \int_1^3 \left(\frac{\sqrt{2+x} - \sqrt{x}}{2+x-x} \right) dx \\ &= \frac{1}{2} \int_1^3 (\sqrt{2+x} - \sqrt{x}) \, dx \\ &= \frac{1}{2} \left[\frac{(2+x)^{3/2}}{3/2} - \frac{x^{3/2}}{3/2} \right]_1^3 \end{aligned}$$

$$\begin{aligned} &= \frac{1}{3} \left[(2+x)^{3/2} - (x)^{3/2} \right]_1^3 \\ &= \frac{1}{3} \left\{ \left[(2+3)^{3/2} - (3)^{3/2} \right] - \left[(2+1)^{3/2} - (1)^{3/2} \right] \right\} \\ &= \frac{1}{3} \left\{ 5^{3/2} - 3^{3/2} - 3^{3/2} + 1^{3/2} \right\} \\ &= \frac{1}{3} \left\{ 5^{3/2} - 2(3)^{3/2} + 1 \right\} \\ \therefore \int_1^3 \frac{1}{\sqrt{2+x} + \sqrt{x}} \, dx &= \frac{1}{3} \left[5^{3/2} - 2(3)^{3/2} + 1 \right] \end{aligned}$$

Ex. 2 : $\int_0^{\pi/2} \sqrt{1 - \cos 4x} \, dx$

Solution : Let $I = \int_0^{\pi/2} \sqrt{1 - \cos 4x} \, dx$

$$\begin{aligned} I &= \int_0^{\pi/2} \sqrt{2 \sin^2 2x} \, dx \\ &\left(\because 1 - \cos A = 2 \sin^2 \frac{A}{2} \right) \\ &= \sqrt{2} \int_0^{\pi/2} \sin 2x \, dx \\ &= \sqrt{2} \left[\frac{-\cos 2x}{2} \right]_0^{\pi/2} \\ &= \frac{\sqrt{2}}{2} \left[\cos 2 \cdot \frac{\pi}{2} - \cos 0 \right] \\ &= -\frac{\sqrt{2}}{2} [\cos \pi - \cos 0] \\ &= -\frac{\sqrt{2}}{2} (-1 - 1) = \sqrt{2} \end{aligned}$$

$\therefore \int_0^{\pi/2} \sqrt{1 - \cos 4x} \, dx = \sqrt{2}$

Ex. 4 : $\int_0^{\pi/4} \frac{\sec^2 x}{2 \tan^2 x + 5 \tan x + 1} \, dx$

Solution : Let $I = \int_0^{\pi/4} \frac{\sec^2 x}{2 \tan^2 x + 5 \tan x + 1} \, dx$

put $\tan x = t \quad \therefore \sec^2 x \, dx = 1 \, dt$

As x varies from 0 to $\frac{\pi}{4}$

t varies from 0 to 1

$$\begin{aligned} &= \int_0^1 \frac{1}{2t^2 + 4t + 1} \, dt \\ &= \frac{1}{2} \int_0^1 \frac{1}{t^2 + 2t + \frac{1}{2}} \, dt \\ &= \frac{1}{2} \int_0^1 \frac{1}{t^2 + 2t + 1 - 1 + \frac{1}{2}} \, dt \\ &= \frac{1}{2} \int_0^1 \frac{1}{(t+1)^2 - \left(\frac{1}{\sqrt{2}}\right)^2} \, dt \end{aligned}$$

Ex. 3 : $\int_0^{\pi/2} \cos^3 x \, dx$

Solution : Let $I = \int_0^{\pi/2} \cos^3 x \, dx$

$$\begin{aligned} &= \int_0^{\pi/2} \frac{1}{4} [\cos 3x + 3 \cos x] \, dx \\ &= \frac{1}{4} \left[\sin 3x \cdot \frac{1}{3} + 3 \sin x \right]_0^{\pi/2} \\ &= \frac{1}{4} \left[\left(\frac{1}{3} \sin 3 \cdot \frac{\pi}{2} + 3 \sin \frac{\pi}{2} \right) - \left(\frac{1}{3} \sin 3(0) + 3 \sin(0) \right) \right] \\ &= \frac{1}{4} \left[\frac{1}{3} \sin \frac{3\pi}{2} + 3 \sin \frac{\pi}{2} - \frac{1}{3} \sin 0 + 3 \sin 0 \right] \\ &= \frac{1}{4} \left[\frac{1}{3} (-1) + 3(1) - 0 \right] \\ &= \frac{1}{4} \left[-\frac{1}{3} + 3 \right] = \frac{1}{4} \left[\frac{8}{3} \right] = \frac{2}{3} \end{aligned}$$

$\therefore \int_0^{\pi/2} \cos^3 x \, dx = \frac{2}{3}$

$$\begin{aligned} &= \frac{1}{2} \frac{1}{2 \left(\frac{1}{\sqrt{2}} \right)} \left[\log \left[\frac{(t+1) - \frac{1}{\sqrt{2}}}{(t+1) + \frac{1}{\sqrt{2}}} \right] \right]_0^1 \\ &= \frac{\sqrt{2}}{4} \log \left[\left(\frac{\sqrt{2}t + \sqrt{2} - 1}{\sqrt{2}t + \sqrt{2} + 1} \right) \right]_0^1 \\ &= \frac{\sqrt{2}}{4} \left[\log \left(\frac{\sqrt{2}(1) + \sqrt{2} - 1}{\sqrt{2}(1) + \sqrt{2} + 1} \right) - \log \left(\frac{\sqrt{2}(0) + \sqrt{2} - 1}{\sqrt{2}(0) + \sqrt{2} + 1} \right) \right] \\ &= \frac{\sqrt{2}}{4} \left[\log \left(\frac{2\sqrt{2} - 1}{2\sqrt{2} + 1} \right) - \log \left(\frac{\sqrt{2} - 1}{\sqrt{2} + 1} \right) \right] \\ &= \frac{\sqrt{2}}{4} \log \left[\left(\frac{2\sqrt{2} - 1}{2\sqrt{2} + 1} \right) \div \left(\frac{\sqrt{2} - 1}{\sqrt{2} + 1} \right) \right] \\ &= \frac{\sqrt{2}}{4} \log \left[\frac{3 + \sqrt{2}}{3 - \sqrt{2}} \right] \end{aligned}$$

Ex. 5 : $\int_1^2 \frac{\log x}{x^2} dx$

Solution : Let $I = \int_1^2 (\log x) \left(\frac{1}{x^2} \right) dx$

$$= \left[(\log x) \int \frac{1}{x^2} dx \right]_1^2 - \int_1^2 \frac{d}{dx} \log x \left(\int \frac{1}{x^2} dx \right) dx$$

$$= \left[(\log x) \left(-\frac{1}{x} \right) \right]_1^2 - \int_1^2 \frac{1}{x} \left(-\frac{1}{x} \right) dx$$

$$= \left[-\frac{1}{x} \log x \right]_1^2 + \int_1^2 \frac{1}{x^2} dx$$

$$= \left[-\frac{1}{x} \log x \right]_1^2 + \left[-\frac{1}{x} \right]_1^2$$

$$= \left[\left(-\frac{1}{2} \log 2 \right) - \left(-\frac{1}{1} \log 1 \right) \right] + \left[\left(-\frac{1}{2} \right) - \left(-\frac{1}{1} \right) \right]$$

$$= -\frac{1}{2} \log 2 - 0 - \frac{1}{2} + 1 = \frac{1}{2} - \frac{1}{2} \log 2 \quad \because \log 1 = 0$$

$\therefore \int_1^2 \frac{\log x}{x^2} dx = \frac{1}{2} (1 - \log 2)$

Ex. 6 : $\int_0^{\pi/2} \frac{\cos x}{1 + \cos x + \sin x} dx$

Solution : Let $I = \int_0^{\pi/2} \frac{\cos x}{1 + \cos x + \sin x} dx$

$$= \int_0^{\pi/2} \frac{\cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)}{2 \cos^2\left(\frac{x}{2}\right) + 2 \sin\left(\frac{x}{2}\right) \cdot \cos\left(\frac{x}{2}\right)} dx$$

$$= \int_0^{\pi/2} \frac{\left[\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right) \right] \left[\cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right) \right]}{2 \left[\cos\left(\frac{x}{2}\right) \right] \left[\cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right) \right]} dx$$

$$= \int_0^{\pi/2} \left[\frac{\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} \right] dx = \int_0^{\pi/2} \left[1 - \tan\left(\frac{x}{2}\right) \right] dx$$

$$= \frac{1}{2} \left[x - \log \left(\sec \frac{x}{2} \right) - \frac{1}{\frac{1}{2}} \right]_0^{\pi/2}$$

$$= \frac{1}{2} \left[\frac{\pi}{2} - 2 \log \left(\sec \frac{\pi}{4} \right) - (0 - 2 \log \sec 0) \right]$$

$$= \frac{1}{2} \left[\frac{\pi}{2} - 2 \log \sqrt{2} - 0 + 2(0) \right] = \frac{1}{2} \left[\frac{\pi}{2} - 2 \log \sqrt{2} \right] = \frac{\pi}{4} - \log \sqrt{2}$$

$$\therefore \int_0^{\pi/2} \frac{\sec^2 x}{1 + \cos x + \sin x} dx = \frac{\pi}{4} - \log \sqrt{2}$$

Ex. 7 : $\int_0^{1/2} \frac{1}{(1-2x^2)\sqrt{1-x^2}} dx$

Solution : Let $I = \int_0^{1/2} \frac{1}{(1-2x^2)\sqrt{1-x^2}} dx$

put $x = \sin \theta \quad \therefore \quad 1 dx = \cos \theta d\theta$

As x varies from 0 to $\frac{1}{2}$, θ varies from 0 to $\frac{\pi}{6}$

$$= \int_0^{\pi/6} \frac{\cos \theta}{(1-2\sin^2 \theta)\sqrt{1-\sin^2 \theta}} d\theta = \int_0^{\pi/6} \frac{\cos \theta}{(\cos 2\theta)\sqrt{\cos^2 \theta}} d\theta$$

$$= \int_0^{\pi/6} \frac{1}{\cos 2\theta} d\theta$$

$$= \int_0^{\pi/6} \sec 2\theta d\theta$$

$$= \left[\log (\sec 2\theta + \tan 2\theta) - \frac{1}{2} \right]_0^{\pi/6}$$

$$= \frac{1}{2} \left[\log \left(\sec 2 \left(\frac{\pi}{6} \right) + \tan 2 \left(\frac{\pi}{6} \right) \right) - \log (\sec 0 + \tan 0) \right]$$

$$= \frac{1}{2} \left[\log \left(\sec \frac{\pi}{3} + \tan \frac{\pi}{3} \right) - \log (1 + 0) \right] \quad \because \log 1 = 0$$

$$= \frac{1}{2} [\log (2 + \sqrt{3}) - 0]$$

$$= \frac{1}{2} \log (2 + \sqrt{3})$$

$$\therefore \int_0^{1/2} \frac{1}{(1-2x^2)\sqrt{1-x^2}} dx = \frac{1}{2} \log (2 + \sqrt{3})$$

Ex. 8 : $\int_0^2 \frac{2^x}{2^x(1+4^x)} dx$

Solution : Let $I = \int_0^2 \frac{2^x}{2^x(1+4^x)} dx$

put $2^x = t \quad \therefore \quad 2^x \cdot \log 2 \, dx = 1 \, dt$

As x varies from 0 to 2, t varies from 1 to 4

$$= \int_1^4 \frac{1}{\log 2} \frac{1}{t(1+t^2)} dt$$

$$= \frac{1}{\log 2} \int_1^4 \frac{1}{t(1+t^2)} dt$$

$$= \frac{1}{\log 2} \int_1^4 \frac{1+t^2-t^2}{t(1+t^2)} dt$$

may be solved by method of partial fraction

$$= \frac{1}{\log 2} \int_1^4 \left[\frac{1+t^2}{t(1+t^2)} - \frac{t^2}{t(1+t^2)} \right] dt$$

$$= \frac{1}{\log 2} \int_1^4 \left[\frac{1}{t} - \frac{t}{1+t^2} \right] dt$$

$$= \frac{1}{\log 2} \left[\int_1^4 \frac{1}{t} dt - \frac{1}{2} \int_1^4 \frac{2t}{1+t^2} dt \right]$$

$$= \frac{1}{\log 2} \left[\log(t) - \frac{1}{2} \log(1+t^2) \right]_1^4$$

$$= \frac{1}{\log 2} \left[\left(\log 4 - \frac{1}{2} \log 17 \right) - \left(\log 1 - \frac{1}{2} \log 2 \right) \right]$$

$$= \frac{1}{\log 2} \left[\log 4 - \frac{1}{2} \log 17 + \frac{1}{2} \log 2 \right]$$

$$\because \log 1 = 0$$

$$= \frac{1}{\log 2} \left[\log \frac{4\sqrt{2}}{\sqrt{17}} \right]$$

$$\therefore \int_0^2 \frac{2^x}{2^x(1+4^x)} dx = \frac{1}{(\log 2)} \left[\log \frac{4\sqrt{2}}{\sqrt{17}} \right]$$

$$= \log_2 \left(\frac{4\sqrt{2}}{\sqrt{17}} \right)$$

Ex. 9 : $\int_{-1}^1 |5x-3| dx$

Solution : Let $I = \int_{-1}^1 |5x-3| dx$

$$|5x-3| = -(5x-3) \text{ for } (5x-3) < 0 \text{ i.e. } x < \frac{3}{5}$$

$$= (5x-3) \text{ for } (5x-3) > 0 \text{ i.e. } x > \frac{3}{5}$$

$$= \int_{-1}^{3/5} |5x-3| dx + \int_{3/5}^1 |5x-3| dx$$

$$= \int_{-1}^{3/5} -(5x-3) dx + \int_{3/5}^1 (5x-3) dx$$

$$= \left[-\left(5 \frac{x^2}{2} - 3x\right) \right]_{-1}^{3/5} + \left[\left(5 \frac{x^2}{2} - 3x\right) \right]_{3/5}^1$$

$$= \left[3x - \frac{5}{2} x^2 \right]_{-1}^{3/5} + \left[\frac{5}{2} x^2 - 3x \right]_{3/5}^1$$

$$= \left[\left(3 \left(\frac{3}{5} \right) - \frac{5}{2} \left(\frac{3}{5} \right)^2 \right) - \left(3(-1) - \frac{5}{2} (-1)^2 \right) \right] + \left[\left(\frac{5}{2} (1)^2 - 3(1) \right) - \left(\frac{5}{2} \left(\frac{3}{5} \right)^2 - 3 \left(\frac{3}{5} \right) \right) \right]$$

$$\begin{aligned}
&= \left[\left(\frac{9}{5} - \frac{9}{10} \right) - \left(-3 - \frac{5}{2} \right) \right] + \left[\left(\frac{5}{2} - 3 \right) - \left(\frac{9}{10} - \frac{9}{5} \right) \right] \\
&= \frac{9}{5} - \frac{9}{10} + 3 + \frac{5}{2} + \frac{5}{2} - 3 - \frac{9}{10} + \frac{9}{5} = 2 \left(\frac{9}{5} - \frac{9}{10} + \frac{5}{2} \right) = 2 \left(\frac{18 - 9 + 25}{5} \right) = \frac{34}{5}
\end{aligned}$$

$$\therefore \int_{-1}^1 |5x - 3| dx = \frac{34}{5}$$

Ex. 10 : $\int_0^{\pi/2} \frac{1}{1 + \sqrt[3]{\tan x}} dx$

Solution : Let $I = \int_0^{\pi/2} \frac{1}{1 + \sqrt[3]{\tan x}} dx$

$$\begin{aligned}
&= \int_0^{\pi/2} \left[\frac{1}{1 + \frac{\sqrt[3]{\sin x}}{\sqrt[3]{\cos x}}} \right] dx \\
&= \int_0^{\pi/2} \frac{\sqrt[3]{\cos x}}{\sqrt[3]{\cos x} + \sqrt[3]{\sin x}} dx \quad \dots (i)
\end{aligned}$$

By property $\int_0^a f(x) dx = \int_0^a f(a-x) dx$

$$\begin{aligned}
I &= \int_0^{\pi/2} \frac{\sqrt[3]{\cos \left(\frac{\pi}{2} - x \right)}}{\sqrt[3]{\cos \left(\frac{\pi}{2} - x \right)} + \sqrt[3]{\sin \left(\frac{\pi}{2} - x \right)}} dx \\
&= \int_0^{\pi/2} \frac{\sqrt[3]{\sin x}}{\sqrt[3]{\sin x} + \sqrt[3]{\cos x}} dx \quad \dots (ii)
\end{aligned}$$

adding (i) and (ii)

$$I + I = \int_0^{\pi/2} \frac{\sqrt[3]{\cos x}}{\sqrt[3]{\cos x} + \sqrt[3]{\sin x}} dx + \int_0^{\pi/2} \frac{\sqrt[3]{\sin x}}{\sqrt[3]{\sin x} + \sqrt[3]{\cos x}} dx$$

$$2I = \int_0^{\pi/2} \frac{\sqrt[3]{\cos x} + \sqrt[3]{\sin x}}{\sqrt[3]{\cos x} + \sqrt[3]{\sin x}} dx$$

$$2I = \int_0^{\pi/2} 1 dx$$

$$I = \frac{1}{2} \left[x \right]_0^{\pi/2} = \frac{1}{2} \left[\frac{\pi}{4} - 0 \right] = \frac{\pi}{4}$$

$$\therefore \int_0^{\pi/2} \frac{1}{1 + \sqrt[3]{\tan x}} dx = \frac{\pi}{4}$$

with the help of the above solved/ illustrative example verify whether the following examples evaluates their definite integrate to be equal to / as $\frac{\pi}{4}$

$$\int_0^{\pi/2} \frac{1}{1 + \cot^3 x} dx ;$$

$$\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx ;$$

$$\int_0^{\pi/2} \frac{\sec x}{\sec x + \operatorname{cosec} x} dx ;$$

$$\int_0^{\pi/2} \frac{\sin^4 x}{\sin^4 x + \cos^4 x} dx ;$$

$$\int_0^{\pi/2} \frac{\operatorname{cosec}^{\frac{5}{2}} x}{\operatorname{cosec}^{\frac{5}{2}} x + \sec^{\frac{5}{2}} x} dx$$

Ex. 11 : $\int_3^8 \frac{(11-x)^2}{x^2 + (1-x)^2} dx$

Solution : Let $I = \int_3^8 \frac{(11-x)^2}{x^2 + (1-x)^2} dx \quad \dots (i)$

By property $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$

$$\begin{aligned} I &= \int_3^8 \frac{[11-(8+3-x)]^2}{[8+3-x]^2 + [11-(8+3-x)]^2} dx = \int_3^8 \frac{[11-(11-x)]^2}{(11-x)^2 + [11-(11-x)]^2} dx \\ &= \int_3^8 \frac{x^2}{(11-x)^2 + x^2} dx \quad \dots (ii) \end{aligned}$$

adding (i) and (ii)

$$I + I = \int_3^8 \frac{(11-x)^2}{x^2 + (1-x)^2} dx + \int_3^8 \frac{x^2}{(11-x)^2 + x^2} dx$$

$$2I = \int_3^8 \frac{(11-x)^2 + x^2}{x^2 + (11-x)^2} dx$$

$$I = \frac{1}{2} \int_3^8 1 dx$$

$$I = \frac{1}{2} \left[x \right]_3^8 = \frac{1}{2} [8-3] = \frac{5}{2}$$

$$\therefore \int_3^8 \frac{(11-x)^2}{x^2 + (1-x)^2} dx = \frac{5}{2}$$

Note that : In general $\int_a^b \frac{f(x)}{f(x) + f(a+b-x)} dx = \frac{1}{2} (b-a)$

verify the generalisation for the following examples :

$$\int_1^2 \frac{\sqrt{x}}{\sqrt{3-x} + \sqrt{x}} dx ;$$

$$\int_2^7 \frac{x^3}{(9-x)^3 + x^3} dx ;$$

$$\int_4^9 \frac{x^{\frac{1}{4}}}{(13-x)^{\frac{1}{4}} + x^{\frac{1}{4}}} dx$$

$$\int_{\pi/6}^{\pi/3} \frac{1}{1 + \sqrt{\cot x}} dx$$

$$\int_{\pi/6}^{\pi/3} \frac{1}{1 + \sqrt{\operatorname{cosec} x}} dx$$

Ex. 12 : $\int_0^{\pi} x \sin^2 x \, dx$

Solution :

Consider, $I = \int_0^{\pi} x \sin^2 x \, dx \dots (i)$

$$I = \int_0^{\pi} (\pi - x) [\sin(\pi - x)]^2 x \, dx$$

$$I = \int_0^{\pi} (\pi - x) \sin^2 x \, dx$$

$$I = \int_0^{\pi} \pi \sin^2 x \, dx - \int_0^{\pi} x \sin^2 x \, dx$$

$$I = \pi \int_0^{\pi} \frac{1}{2} (1 - \cos 2x) \cdot dx - I \dots \text{by (i)}$$

$$I + I = \frac{\pi}{2} \int_0^{\pi} (1 - \cos 2x) \, dx$$

$$2I = \frac{\pi}{2} \left[x - \sin 2x \cdot \frac{1}{2} \right]_0^{\pi}$$

$$I = \frac{\pi}{4} \left[\left(\pi - \frac{1}{2} \sin 2\pi \right) - \left(0 - \frac{1}{2} \sin 0 \right) \right]$$

$$= \frac{\pi}{4} [\pi] \quad \because \sin 0 = 0; \sin 2\pi = 0$$

$$= \frac{\pi^2}{4}$$

$$\therefore \int_0^{\pi} x^2 \cdot \sin^2 x \, dx = \frac{\pi^2}{4}$$

Ex. 13 : Evaluate the integral $\int_0^{\pi} \cos^2 x \, dx$ using the result/ property.

Solution :

$$\int_0^{2a} f(x) \, dx = \int_0^a f(x) \, dx + \int_0^a f(2a - x) \, dx$$

Let, $I = \int_0^{\pi} \cos^2 x \, dx$

$$= \int_0^{2\left(\frac{\pi}{2}\right)} \cos^2 x \, dx$$

$$= \int_0^{\pi/2} \cos^2 x \, dx + \int_0^{\pi/2} \left[\cos \left(2\frac{\pi}{2} - x \right) \right]^2 dx$$

$$= \int_0^{\pi/2} \cos^2 x \, dx + \int_0^{\pi/2} \cos^2 x \, dx$$

$$\because \cos(\pi - x) = -\cos x$$

$$= 2 \cdot \int_0^{\pi/2} \cos^2 x \, dx$$

$$= \int_0^{\pi/2} (1 + \cos 2x) \, dx$$

$$= \left[x + \sin 2x \cdot \frac{1}{2} \right]_0^{\pi/2}$$

$$= \left[\left(\frac{\pi}{2} + \frac{1}{2} \sin 2\frac{\pi}{2} \right) - \left(0 + \frac{1}{2} \sin 2(0) \right) \right]$$

$$= \frac{\pi}{2} + 0 \quad \because \sin 0 = 0; \sin \pi = 0$$

$$= \frac{\pi}{2}$$

$$\therefore \int_0^{\pi} \cos^2 x \, dx = \frac{\pi}{2}$$

Ex. 14 : $\int_{-\pi}^{\pi} \frac{x(1 + \sin x)}{1 + \cos^2 x} dx$

Solution : Let $I = \int_{-\pi}^{\pi} \frac{x(1 + \sin x)}{1 + \cos^2 x} dx$

$$= \left[\left(\int_{-\pi}^{\pi} \frac{x}{1 + \cos^2 x} dx \right) + \left(\int_{-\pi}^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx \right) \right]$$

The function $\frac{x}{1 + \cos^2 x}$ is odd function and the function $\frac{x \sin x}{1 + \cos^2 x}$ is even function.

$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx, \text{ if } f(x) \text{ even function}$$

$$= 0, \text{ if } f(x) \text{ is odd function}$$

$$\therefore I = 0 + 2 \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$$

$$\therefore I = 2 \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx \quad \dots (i)$$

$$= 2 \int_0^{\pi} \frac{(\pi - x) \sin(\pi - x)}{1 + [\cos(\pi - x)]^2} dx$$

$$= 2 \int_0^{\pi} \frac{(\pi - x) \sin x}{1 + (-\cos x)^2} dx$$

$$= 2\pi \int_0^{\pi} \frac{\pi \sin x - x \sin x}{1 + \cos^2 x} dx$$

$$= 2\pi \int_0^{\pi} \frac{\sin x dx}{1 + \cos^2 x} - 2 \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$$

$$I = 2\pi \int_0^{\pi} \frac{\sin x dx}{1 + \cos^2 x} - I \dots \text{by eq.(i)}$$

$$I + I = 2\pi \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx \quad \dots (ii)$$

put $\cos x = t \quad \therefore -\sin x dx = +dt$

As x varies from 0 to π , t varies from 1 to -1

$$2I = 2\pi \int_{-1}^1 \frac{-1}{1 + t^2} dt$$

$$I = \pi \cdot 2 \int_0^1 \frac{1}{1 + t^2} dt \quad \left(\text{where } \frac{1}{1 + t^2} \text{ is even function.} \right)$$

$$\begin{aligned}
 I &= 2\pi \left[\tan^{-1} t \right]_0^1 \\
 &= 2\pi [\tan^{-1}(1) - \tan^{-1}(0)] \\
 &= 2\pi \left(\frac{\pi}{4} - 0 \right) = \frac{\pi^2}{2}
 \end{aligned}$$

$$\therefore \int_{-\pi}^{\pi} \frac{x(1 + \sin x)}{1 + \cos^2 x} dx = \frac{\pi^2}{2}$$

Ex. 15 : $\int_0^3 x [x] dx$, where $[x]$ denote greatest integrate function not greater than x .

Solution : Let $I = \int_0^3 x [x] dx$

$$\begin{aligned}
 I &= \int_0^1 x [x] dx + \int_1^2 x [x] dx + \int_2^3 x [x] dx \\
 &= \int_0^1 x(0) dx + \int_1^2 x(1) dx + \int_2^3 x(2) dx \\
 &= 0 + \left[\frac{x^2}{2} \right]_1^2 + \left[x^2 \right]_2^3 \\
 &= 0 + \left(\frac{4}{2} - \frac{1}{2} \right) + (9 - 4) \\
 &= \frac{3}{2} + 5 = \frac{13}{2}
 \end{aligned}$$

$$\therefore \int_0^3 x [x] dx = \frac{13}{2}$$

EXERCISE 4.2

I. Evaluate :

- | | | | |
|---|---|---|--|
| (1) $\int_1^9 \frac{x+1}{\sqrt{x}} dx$ | (2) $\int_2^3 \frac{1}{x^2 + 5x + 6} dx$ | (8) $\int_0^{\pi/4} \sqrt{1 + \sin 2x} dx$ | (9) $\int_0^{\pi/4} \sin^4 x dx$ |
| (3) $\int_0^{\pi/4} \cot^2 x$ | (4) $\int_{-\pi/4}^{\pi/4} \frac{1}{1 - \sin x} dx$ | (10) $\int_{-4}^2 \frac{1}{x^2 + 4x + 13} dx$ | (11) $\int_0^4 \frac{1}{\sqrt{4x - x^2}} dx$ |
| (5) $\int_3^5 \frac{1}{\sqrt{2x+3} - \sqrt{2x-3}} dx$ | | (12) $\int_0^1 \frac{1}{\sqrt{3+2x-x^2}} dx$ | (13) $\int_0^{\pi/2} x \sin x dx$ |
| (6) $\int_0^1 \frac{x^2-2}{x^2+1} dx$ | (7) $\int_0^{\pi/4} \sin 4x \sin 3x dx$ | (14) $\int_0^1 x \tan^{-1} x dx$ | (15) $\int_0^{\infty} x e^{-x} dx$ |

II. Evaluate :

- (1) $\int_0^{\frac{1}{\sqrt{2}}} \frac{\sin^{-1} x}{(1-x^2)^{\frac{3}{2}}} dx$
- (2) $\int_0^{\pi/4} \frac{\sec^2 x}{3 \tan^2 x + 4 \tan x + 1} dx$
- (3) $\int_0^{\pi/4} \frac{\sin 2x}{\sin^4 x + \cos^4 x} dx$
- (4) $\int_0^{\pi/2} \sqrt{\cos x} \sin^3 x \, dx$
- (5) $\int_0^{\pi/2} \frac{1}{5 + 4 \cos x} dx$
- (6) $\int_0^{\pi/4} \frac{\cos x}{4 - \sin^2 x} dx$
- (7) $\int_0^{\pi/2} \frac{\cos x}{(1 + \sin x)(2 + \sin x)} dx$
- (8) $\int_{-1}^1 \frac{1}{a^2 e^x + b^2 e^{-x}} dx$
- (9) $\int_0^{\pi} \frac{1}{3 + 2 \sin x + \cos x} dx$
- (10) $\int_0^{\pi/4} \sec^4 x \, dx$
- (11) $\int_0^1 \sqrt{\frac{1-x}{1+x}} dx$
- (12) $\int_0^{\pi} \sin^3 x (1 + 2 \cos x) (1 + \cos x)^2 \, dx$
- (13) $\int_0^{\pi/2} \sin 2x \tan^{-1}(\sin x) \, dx$
- (14) $\int_{\frac{1}{\sqrt{2}}}^1 \frac{(e^{\cos^{-1} x})(\sin^{-1} x)}{\sqrt{1-x^2}} dx$
- (15) $\int_2^3 \frac{\cos(\log x)}{x} \cdot dx$

III. Evaluate :

- (1) $\int_0^a \frac{1}{x + \sqrt{a^2 - x^2}} dx$
- (2) $\int_0^{\pi/2} \log \tan x \, dx$
- (3) $\int_0^1 \log \left(\frac{1}{x} - 1 \right) dx$
- (4) $\int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cdot \cos x} dx$
- (5) $\int_0^3 x^2 (3-x)^{\frac{5}{2}} dx$
- (6) $\int_{-3}^3 \frac{x^3}{9-x^2} dx$
- (7) $\int_{-\pi/2}^{\pi/2} \log \left(\frac{2 + \sin x}{2 - \sin x} \right) dx$
- (8) $\int_{-\pi/4}^{\pi/4} \frac{x + \frac{\pi}{4}}{2 - \cos 2x} dx$
- (9) $\int_{-\pi/4}^{\pi/4} x^3 \sin^4 x \, dx$
- (10) $\int_0^1 \frac{\log(x+1)}{x^2+1} dx$
- (11) $\int_{-1}^1 \frac{x^3+2}{\sqrt{x^2+4}} dx$
- (12) $\int_{-a}^a \frac{x+x^3}{16-x^2} dx$
- (13) $\int_0^1 t^2 \sqrt{1-t} \, dt$
- (14) $\int_0^{\pi} x \sin x \cos^2 x \, dx$
- (15) $\int_0^1 \frac{\log x}{\sqrt{1-x^2}} dx$

Note that :

To evaluate the integrals of the type $\int_0^{\pi/2} \sin^n x \, dx$ and $\int_0^{\pi/2} \cos^n x \, dx$, the results used are known as

'reduction formulae' which are stated as follows :

$$\int_0^{\pi/2} \sin^n x \, dx = \frac{(n-1)}{n} \cdot \frac{(n-3)}{(n-2)} \cdot \frac{(n-5)}{(n-4)} \dots \frac{4}{5} \cdot \frac{2}{3}, \quad \text{if } n \text{ is odd.}$$

$$= \frac{(n-1)}{n} \cdot \frac{(n-3)}{(n-2)} \cdot \frac{(n-5)}{(n-4)} \dots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}, \quad \text{if } n \text{ is even.}$$

$$\int_0^{\pi/2} \cos^n x \, dx = \int_0^{\pi/2} \left[\cos \left(\frac{\pi}{2} - x \right) \right]^n dx \quad \dots \text{by property}$$

$$= \int_0^{\pi/2} [\sin x]^n dx$$

$$= \int_0^{\pi/2} \sin^n x \, dx$$

$$\int_0^{\pi/2} \sin^7 x \, dx = \frac{(7-1)}{7} \cdot \frac{(7-3)}{(7-2)} \cdot \frac{(7-5)}{(7-4)}$$

$$= \frac{(7-1) \cdot (7-3) \cdot (7-5)}{7 \cdot (7-2) \cdot (7-4)}$$

$$= \frac{6 \cdot 4 \cdot 2}{7 \cdot 5 \cdot 3} = \frac{16}{35}$$

$$\int_0^{\pi/2} \cos^8 x \, dx = \frac{(8-1)}{8} \cdot \frac{(8-3)}{(8-2)} \cdot \frac{(8-5)}{(8-4)} \cdot \frac{(8-7)}{(8-6)} \cdot \frac{\pi}{2}$$

$$= \frac{(8-1) \cdot (8-3) \cdot (8-5) \cdot (8-7)}{8 \cdot (8-2) \cdot (8-4) \cdot (8-6)} \cdot \frac{\pi}{2}$$

$$= \frac{7 \cdot 5 \cdot 3 \cdot 1}{8 \cdot 6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2}$$

$$= \frac{35\pi}{256}$$



Let us Remember

$$\sum_{r=0}^{n-1} (x_{r+1} - x_r) \cdot f(t_r) = \sum_{r=0}^{n-1} g(x_{r+1}) - g(x_r) = g(b) - g(a)$$

Thus taking limit as $n \rightarrow \infty$

$$g(b) - g(a) = \lim_{n \rightarrow \infty} \sum (x_{r+1} - x_r) \cdot f(t_r) = \lim_{n \rightarrow \infty} S_n = \int_a^b f(x) dx$$

$$\text{Fundamental theorem of integral calculus : } \int_a^b f(x) dx = g(b) - g(a)$$

$$\text{Property I : } \int_a^a f(x) dx = 0$$

$$\text{Property II : } \int_a^b f(x) dx = - \int_b^a f(x) dx$$

$$\text{Property III : } \int_a^b f(x) dx = \int_a^b f(t) dt$$

$$\text{Property IV : } \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx \quad \text{where } a < c < b \text{ i.e. } c \in [a, b]$$

$$\text{Property V : } \int_a^b f(x) dx = \int_a^b f(a+b-x) dx$$

$$\text{Property VI : } \int_0^a f(x) dx = \int_0^a f(a-x) dx$$

$$\text{Property VII : } \int_0^{2a} f(x) dx = \int_0^a f(x) dx + \int_0^a f(2a-x) dx$$

$$\text{Property VIII : } \int_{-a}^a f(x) dx = 2 \cdot \int_0^a f(x) dx, \text{ if } f(x) \text{ even function}$$

$$= 0, \text{ if } f(x) \text{ is odd function}$$

$f(x)$ even function if $f(-x) = f(x)$ and $f(x)$ odd function if $f(-x) = -f(x)$

✿ 'Reduction formulae' which are stated as follows :

$$\int_0^{\pi/2} \sin^n x dx = \frac{(n-1)}{n} \cdot \frac{(n-3)}{(n-2)} \cdot \frac{(n-5)}{(n-4)} \cdots \frac{4}{5} \cdot \frac{2}{3}, \quad \text{if } n \text{ is odd.}$$

$$= \frac{(n-1)}{n} \cdot \frac{(n-3)}{(n-2)} \cdot \frac{(n-5)}{(n-4)} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}, \quad \text{if } n \text{ is even.}$$

$$\int_0^{\pi/2} \cos^n x dx = \int_0^{\pi/2} \left[\cos \left(\frac{\pi}{2} - 0 \right) \right]^n dx = \int_0^{\pi/2} [\sin x]^n dx = \int_0^{\pi/2} \sin^n x dx$$

MISCELLANEOUS EXERCISE 4

(I) Choose the correct option from the given alternatives :

- (1) $\int_2^3 \frac{dx}{x(x^3-1)} =$
 (A) $\frac{1}{3} \log \left(\frac{208}{189} \right)$ (B) $\frac{1}{3} \log \left(\frac{189}{208} \right)$ (C) $\log \left(\frac{208}{189} \right)$ (D) $\log \left(\frac{189}{208} \right)$
- (2) $\int_0^{\pi/2} \frac{\sin^2 x \, dx}{(1 + \cos x)^2} =$
 (A) $\frac{4 - \pi}{2}$ (B) $\frac{\pi - 4}{2}$ (C) $4 - \frac{\pi}{2}$ (D) $\frac{4 + \pi}{2}$
- (3) $\int_0^{\log 5} \frac{e^x \sqrt{e^x - 1}}{e^x + 3} dx =$
 (A) $3 + 2\pi$ (B) $4 - \pi$ (C) $2 + \pi$ (D) $4 + \pi$
- (4) $\int_0^{\pi/2} \sin^6 x \cos^2 x \, dx =$
 (A) $\frac{7\pi}{256}$ (B) $\frac{3\pi}{256}$ (C) $\frac{5\pi}{256}$ (D) $\frac{-5\pi}{256}$
- (5) If $\int_0^1 \frac{dx}{\sqrt{1+x} - \sqrt{x}} = \frac{k}{3}$, then k is equal to
 (A) $\sqrt{2} (2\sqrt{2} - 2)$ (B) $\frac{\sqrt{2}}{3} (2 - 2\sqrt{2})$ (C) $\frac{2\sqrt{2} - 2}{3}$ (D) $4\sqrt{2}$
- (6) $\int_1^2 \frac{1}{x^2} e^{\frac{1}{x}} dx =$
 (A) $\sqrt{e} + 1$ (B) $\sqrt{e} - 1$ (C) $\sqrt{e} (\sqrt{e} - 1)$ (D) $\frac{\sqrt{e} - 1}{e}$
- (7) If $\int_2^e \left[\frac{1}{\log x} - \frac{1}{(\log x)^2} \right] dx = a + \frac{b}{\log 2}$, then
 (A) $a = e, b = -2$ (B) $a = e, b = 2$ (C) $a = -e, b = 2$ (D) $a = -e, b = -2$
- (8) Let $I_1 = \int_e^{e^2} \frac{dx}{\log x}$ and $I_2 = \int_1^2 \frac{e^x}{x} dx$, then
 (A) $I_1 = \frac{1}{3} I_2$ (B) $I_1 + I_2 = 0$ (C) $I_1 = 2I_2$ (D) $I_1 = I_2$

$$(9) \int_0^9 \frac{\sqrt{x}}{\sqrt{x} + \sqrt{9-x}} dx =$$

- (A) 9 (B) $\frac{9}{2}$ (C) 0 (D) 1

$$(10) \text{ The value of } \int_{-\pi/4}^{\pi/4} \log \left(\frac{2 + \sin \theta}{2 - \sin \theta} \right) d\theta \text{ is}$$

- (A) 0 (B) 1 (C) 2 (D) π

(II) Evaluate the following :

$$(1) \int_0^{\pi/2} \frac{\cos x}{3 \cos x + \sin x} dx$$

$$(2) \int_{\pi/4}^{\pi/2} \frac{\cos \theta}{\left[\cos \frac{\theta}{2} + \sin \frac{\theta}{2} \right]^3} d\theta$$

$$(3) \int_0^1 \frac{1}{1 + \sqrt{x}} dx$$

$$(4) \int_0^{\pi/4} \frac{\tan^3 x}{1 + \cos 2x} dx$$

$$(5) \int_0^1 t^5 \sqrt{1-t^2} dt$$

$$(6) \int_0^1 (\cos^{-1} x)^2 dx$$

$$(7) \int_{-1}^1 \frac{1+x^3}{9-x^2} dx$$

$$(8) \int_0^{\pi} x \sin x \cos^4 x dx$$

$$(9) \int_0^{\pi} \frac{x}{1 + \sin^2 x} dx$$

$$(10) \int_1^{\infty} \frac{1}{\sqrt{x}(1+x)} dx$$

(III) Evaluate :

$$(1) \int_0^1 \left(\frac{1}{1+x^2} \right) \sin^{-1} \left(\frac{2x}{1+x^2} \right) dx$$

$$(2) \int_0^{\pi/2} \frac{1}{6 - \cos x} dx$$

$$(3) \int_0^a \frac{1}{a^2 + ax - x^2} dx$$

$$(4) \int_{\pi/5}^{3\pi/10} \frac{\sin x}{\sin x + \cos x} dx$$

$$(5) \int_0^1 \sin^{-1} \left(\frac{2x}{1+x^2} \right) dx$$

$$(6) \int_0^{\pi/4} \frac{\cos 2x}{1 + \cos 2x + \sin 2x} dx$$

$$(7) \int_0^{\pi/2} (2 \log \sin x - \log \sin 2x) dx$$

$$(8) \int_0^{\pi} (\sin^{-1} x + \cos^{-1} x)^3 \sin^3 x dx$$

$$(9) \int_0^4 [\sqrt{x^2 + 2x + 3}]^{-1} dx$$

$$(10) \int_{-2}^3 |x-2| dx$$

(IV) Evaluate the following :

(1) If $\int_0^a \sqrt{x} \, dx = 2a \int_0^{\pi/2} \sin^3 x \, dx$ then find the value of $\int_a^{a+1} x \, dx$

(2) If $\int_0^k \frac{1}{2+8x^2} \, dx = \frac{\pi}{16}$ Find k .

(3) If $f(x) = a + bx + cx^2$, show that $\int_0^1 f(x) \, dx = \frac{1}{6} \left[f(0) + 4f\left(\frac{1}{2}\right) + f(1) \right]$.

